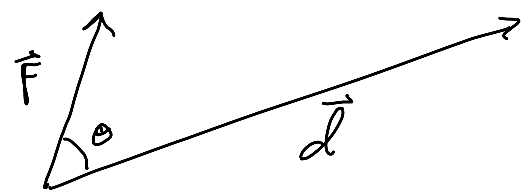
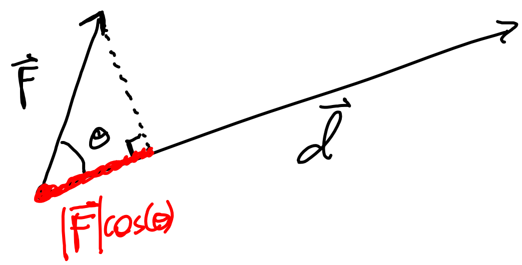
Section 2.3 Dot product.

Given two vectors $\vec{F}$ and $\vec{d}$ with angle $0 \le \theta \le 180°$ between their common tail endpoint



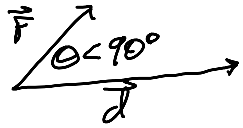
define the dot product $\vec{F} \cdot \vec{d} = |\vec{F}|\,|\vec{d}|\cos(\theta)$.

Notice that $|F|\cos(\theta)$ is the length in the direction of $\vec{d}$ shown below.

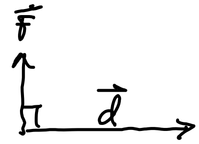


So $\vec{F} \cdot \vec{d} = |\vec{d}|\,|\vec{F}|\cos(\theta)$ is the component of $\vec{F}$ in the direction of $\vec{d}$ times $|\vec{d}|$.

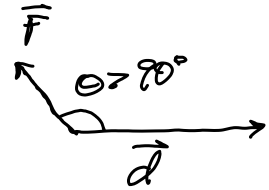
Since force $\times$ distance $=$ work when force and distance are parallel vectors, $\vec{F} \cdot \vec{d}$ represents work done by a force $\vec{F}$ along displacement vector $\vec{d}$ when they are not parallel.



$\vec{F} \cdot \vec{d} > 0$



$\vec{F} \cdot \vec{d} = 0$

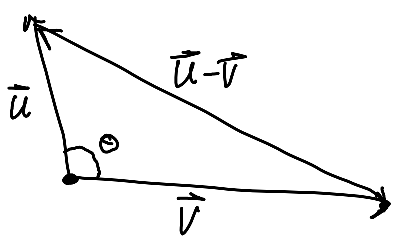


$\vec{F} \cdot \vec{d} < 0$

<u>Theorem</u> If $\vec{u} = \langle a, b, c \rangle$ $\vec{v} = \langle x, y, z \rangle$, Then $\vec{u} \cdot \vec{v} = ax + by + cz$

If $\vec{u} = \langle a, b \rangle$ $\vec{v} = \langle x\ y \rangle$, Then $\vec{u} \cdot \vec{v} = ax + by$

<u>proof</u> We'll do the proof for 2D only.



Assuming That The law of cosines is true

$$|\vec{u} - \vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos(\theta)$$

This is how we defined $\vec{u} \cdot \vec{v}$

$$|\langle a, b \rangle - \langle x, y \rangle|^2 = (a^2 + b^2) + (x^2 + y^2) - 2\vec{u} \cdot \vec{v}$$

$$|\langle a - x, b - y \rangle|^2 = a^2 + b^2 + x^2 + y^2 - 2\vec{u} \cdot \vec{v}$$

$$(a - x)^2 + (b - y)^2 = a^2 + b^2 + x^2 + y^2 - 2\vec{u} \cdot \vec{v}$$

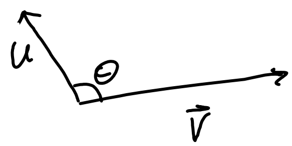
$$\cancel{a^2} - 2ax + \cancel{x^2} + \cancel{b^2} - 2by + \cancel{y^2} = \cancel{a^2} + \cancel{b^2} + \cancel{x^2} + \cancel{y^2} - 2\vec{u} \cdot \vec{v}$$

$$-2ax - 2by = -2u \cdot \vec{v}$$

$$ax + by = \vec{u} \cdot \vec{v}$$

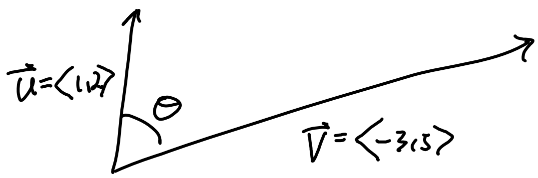
Done ■

A nice cosequence of This result is The following



$$\cos(\theta) = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}$$

example



Find θ.

$$\cos(\theta) = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|} = \frac{(1)(-3) + (2)(5)}{\sqrt{5}\ \sqrt{34}} = \frac{7}{\sqrt{170}}$$

$$\theta = \operatorname{Arccos}\left(\frac{7}{\sqrt{170}}\right) \approx 57.53^\circ$$

example Let $\vec{u} = \langle 1, 2, -3 \rangle$ $\vec{v} = \langle 2, 1, t \rangle$ Find t so That $\vec{u}$ and $\vec{v}$ are perpendicular.

Because $\vec{u}$ is perpendicular to $\vec{v}$, need $\vec{u} \cdot \vec{v} = 0$.

$$0 = \vec{u} \cdot \vec{v} = 2 + 2 - 3t$$

$$0 = 4 - 3t$$

$$t = \frac{4}{3}$$

Basic algebraic properties

1. $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$

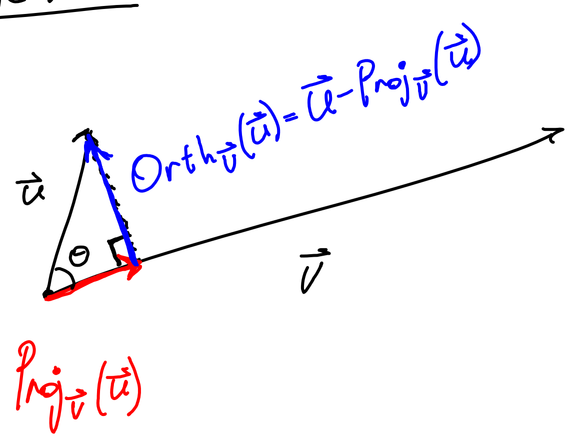
2. $\vec{u}\cdot\vec{u} = |\vec{u}|^2$ $\longrightarrow$ ~~proof~~ If $\vec{u} = \langle a, b, c\rangle$ Then $|\vec{u}| = \sqrt{a^2+b^2+c^2}$ so $|u|^2 = a^2+b^2+c^2 = \vec{u}\cdot\vec{u}$

3. $\vec{u}\cdot\vec{0} = 0$

4. $\vec{u}\cdot(k\vec{v}) = k(\vec{u}\cdot\vec{v})$

5. $\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w}$.

Projections



Note That $|Proj_{\vec{v}}(\vec{u})| = |\vec{u}|\cos(\theta)$

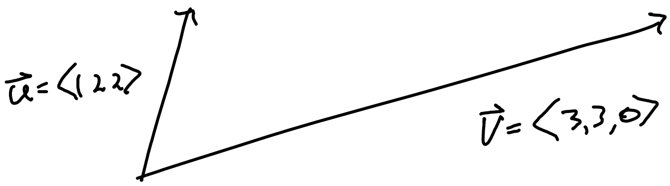
Also, the direction of $Proj_{\vec{v}}(\vec{u})$ is $\frac{1}{|\vec{v}|}\vec{v}$

Therefore $Proj_{\vec{v}}(\vec{u}) = |\vec{u}|\cos(\theta)\left(\frac{1}{|\vec{v}|}\vec{v}\right) = \frac{|\vec{u}|\cos(\theta)}{|\vec{v}|}\vec{v} = \frac{|\vec{u}||\vec{v}|\cos(\theta)}{|\vec{v}|^2}\vec{v}$

Thus $\boxed{Proj_{\vec{v}}(\vec{u}) = \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v}}$

Also $\boxed{Orth_{\vec{v}}(\vec{u}) = \vec{u} - \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v}}$

example



$$\mathrm{Proj}_{\vec{v}}(\vec{u}) = \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v} = \frac{3+6+0}{9+9+0}\langle 3,3,0\rangle = \left\langle \frac{3}{2}, \frac{3}{2}, 0 \right\rangle$$

$$\mathrm{Orth}_{\vec{v}}(\vec{u}) = \vec{u} - \left\langle \frac{3}{2}, \frac{3}{2}, 0 \right\rangle = \left\langle -\frac{1}{2}, \frac{1}{2}, 2 \right\rangle$$

Notice $\langle 3,3,0\rangle \cdot \left\langle -\frac{1}{2}, \frac{1}{2}, 0 \right\rangle = -\frac{3}{2} + \frac{3}{2} + 0 = 0.$