Discussion Problems for our next meeting

Sections 5.3, 5.4 on the syllabus

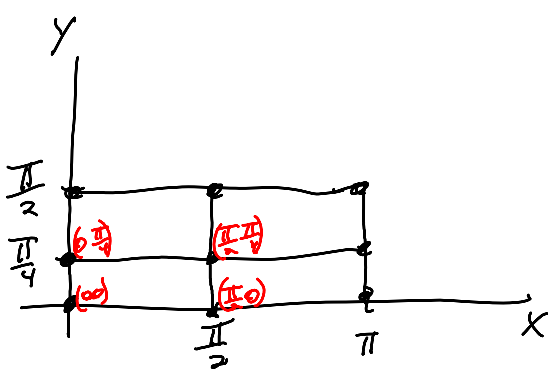
Exam 3 covers sections 5.1 - 5.6

scheduled for Wednesday 11/2

Sol

(4) $f(x,y) = \cos(x) + \sin(y)$ Estimate $\displaystyle\iint_R \sin(x)+\cos(y)\, dA$

Where R: $0 \le x \le \pi$, $0 \le y \le \frac{\pi}{2}$ with a Riemann sum using $m=n=2$ subdivisions.



$\Delta x = \frac{\pi}{2}$

$\Delta y = \frac{\pi}{4}$

$\Delta A = \Delta x \Delta y = \frac{\pi^2}{8}$

$$\left(f(0,0) + f\left(\tfrac{\pi}{2},0\right) + f\left(0,\tfrac{\pi}{4}\right) + f\left(\tfrac{\pi}{2},\tfrac{\pi}{4}\right)\right)\Delta A =$$

$$\left(1 + 0 + \left(1+\tfrac{\sqrt{2}}{2}\right) + \left(0+\tfrac{\sqrt{2}}{2}\right)\right)\frac{\pi^2}{8} =$$

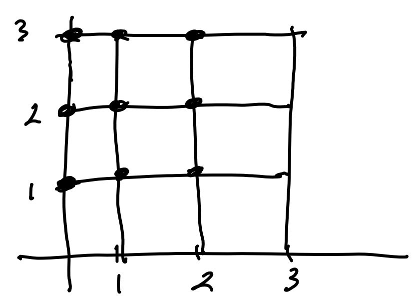
$$\left(2+\sqrt{2}\right)\frac{\pi^2}{8} \approx 4.2121$$

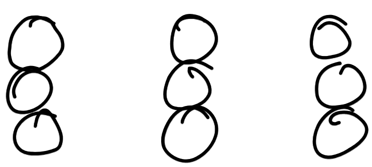
Let's calculate the true value

$$\iint_R \sin(x)+\cos(y)\,dA = \int_0^{\pi}\int_0^{\frac{\pi}{2}} \cos(x)+\sin(y)\,dy\,dx = \int_0^{\pi} y\cos(x)-\cos(y)\Big]_{y=0}^{y=\frac{\pi}{2}}\,dx$$

$$= \int_0^{\pi} \frac{\pi}{2}\cos(x)-(0-1)\,dx = \int_0^{\pi}\frac{\pi}{2}\cos(x)+1\,dx = \frac{\pi}{2}\sin(x)+x\Big]_{x=0}^{x=\pi}$$

$$= \pi \approx 3.14159$$





sample points

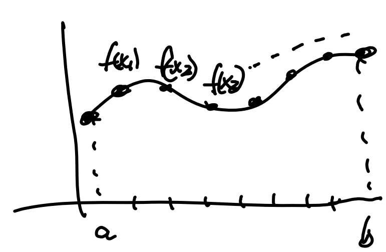
$\Delta x = \Delta y = 1$ so $\Delta A = 1\cdot 1 = 1$

$$\text{Volume} \approx (6.5+6.4+6+7+7.5+6.5+6.7+6.5+6)\cdot 1$$

$$= 59.6 \text{ ft}^3$$

The average value of a single-variable function $f(x)$ on $a \leq x \leq b$ is

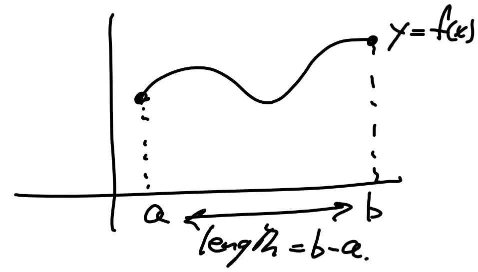
$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} f(x_i)$$



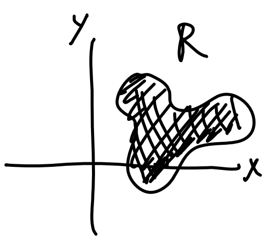
$$\frac{b-a}{n} = \Delta x$$

$$= \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} f(x_i)\frac{\Delta x}{\Delta x} = \lim_{n\to\infty}\left(\sum_{i=1}^{n} f(x_i)\Delta x\right)\frac{1}{n}\frac{1}{\Delta x} = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x\,\frac{1}{\cancel{n}}\frac{\cancel{n}}{b-a}$$

$$= \frac{1}{b-a}\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x \underset{\text{by definition}}{=} \boxed{\frac{1}{b-a}\int_a^b f(x)\,dx}$$

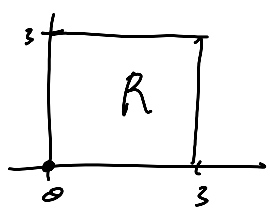


Similarly, the average value of a 2-variable function on a region R is



$$\frac{1}{\text{Area}(R)}\iint_R f(x,y)\,dA$$

So in our example #8, the average value of $f(x,y)$ on



is

$$\frac{1}{\text{Area}(R)}\iint_R f(x,y)\,dA \approx \frac{1}{9}(59.6) = 6.6222\ldots$$

14) $$\int_0^2\int_0^1 (x+2e^y-3)\,dx\,dy = \int_0^1\int_0^2 (x+2e^y-3)\,dy\,dx =$$

$$\int_0^1 \left. xy+2e^y-3y \right]_{y=0}^{y=2} dx = \int_0^1 2x+2e^2-6-(0+2-0)\,dx$$

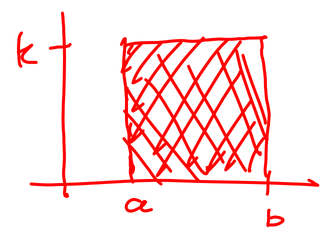
$$= \int_0^1 2x+(2e^2-8)\,dx = \left. x^2+(2e^2-8)x \right]_0^1 = 1+2e^2-8 = \boxed{2e^2-7}$$

18) omit

24) omit

26) $$\int_1^e\int_1^2 \underbrace{x^2\ln(x)}_{\text{constant with respect to } y.}\,dy\,dx =$$

Remember



$$\int_a^b k\,dx = (b-a)k$$

$$\int_1^e x^2\ln(x)\,dx = \left. \frac{1}{3}x^3\ln(x) \right]_{x=1}^{x=e} - \int_1^e \frac{1}{3}x^2\,dx = \left(\frac{e^3}{3}-0\right) - \left. \frac{1}{9}x^3 \right]_{x=1}^{x=e}$$

$u=\ln(x) \quad du=\frac{1}{x}dx$

$dv=x^2dx \quad v=\frac{1}{3}x^3$

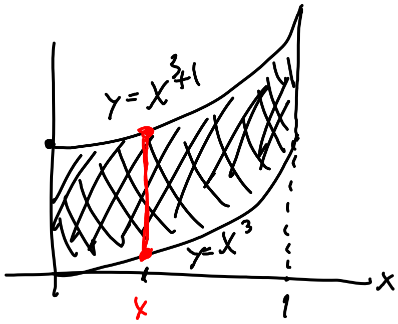
$$= \frac{e^3}{3} - \left(\frac{e^3}{9}-\frac{1}{9}\right)$$

$$= \boxed{\frac{1+2e^3}{9}}$$

30, 34, 38 omit.

5.2

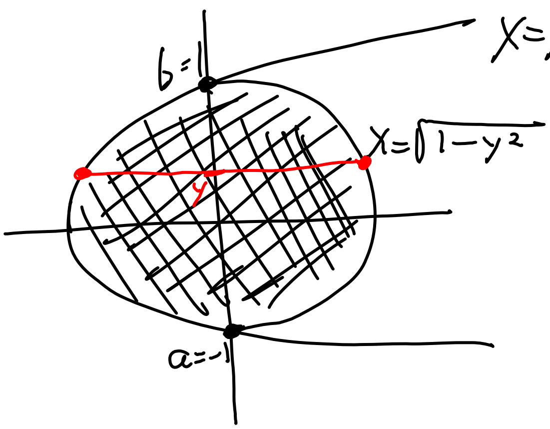
(60)



Type 1

$$\boxed{\begin{gathered} 0 \le x \le 1 \\ x^3 \le y \le x^3+1 \end{gathered}}$$

(66)



Type 2

$$\boxed{\begin{gathered} -1 \le y \le 1 \\ y^2-1 \le x \le \sqrt{1-y^2} \end{gathered}}$$

$y^2-1=\sqrt{1-y^2}$

$(y^2-1)^2=1-y^2$

$y^4-2y^2+\cancel{1}=\cancel{1}-y^2$

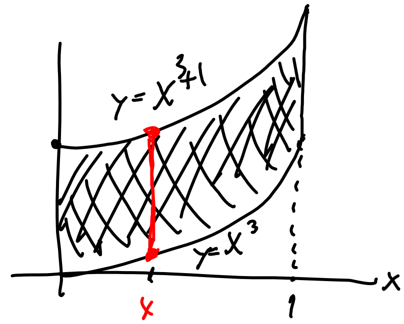
$y^4-y^2=0$

$y^2(y^2-1)=0$

$y^2(y-1)(y+1)=0$

$y=0$ or $y=1$ or $y=-1$

(74) Calculate $\iint\limits_D 2x+5y\,dA$

$D =$ 

Type 1

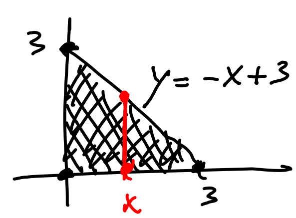
$$\boxed{\begin{array}{l} 0 \le x \le 1 \\ x^3 \le y \le x^3+1 \end{array}}$$

$$\iint\limits_D 2x+5y\,dA = \int_0^1\int_{x^3}^{x^3+1} (2x+5y)\,dy\,dx = \int_0^1 2xy + \frac{5}{2}y^2\Big]_{y=x^3}^{y=x^3+1} dx$$

$$= \int_0^1 2x(\cancel{x^3}+1) + \frac{5}{2}(x^3+1)^2 - \cancel{2x(x^3)} - \frac{5}{2}(x^3)^2\,dx$$

$$= \int_0^1 2x + \frac{5}{2}(\cancel{x^6}+2x^3+1) - \cancel{\frac{5}{2}x^6}\,dx = \int_0^1 2x + 5x^3 + \frac{5}{2}\,dx$$

$$= x^2 + \frac{5}{4}x^4 + \frac{5}{2}x\Big]_{x=0}^{x=1} = 1 + \frac{5}{4} + \frac{5}{2} = \frac{19}{4}$$

(78) Calculate $\iint\limits_R \sin(y)\,dA$ where R 

Type 1

$$\boxed{\begin{array}{l} 0 \le x \le 3 \\ 0 \le y \le 3-x \end{array}}$$

$$\iint\limits_R \sin(y)\,dA = \int_0^3\int_0^{3-x} \sin(y)\,dy\,dx = \int_0^3 -\cos(y)\Big]_{y=0}^{y=3-x} dx = \int_0^3 -\cos(3-x)+1\,dx$$

$$= \int_0^3 \underbrace{1-\cos(3-x)\,dx}_{\text{let } u=3-x} = x + \sin(3-x)\Big]_{x=0}^{x=3} = 3+\sin(0) - \left(0+\sin(3)\right)$$

$$= \boxed{3 - \sin(3)}$$

8

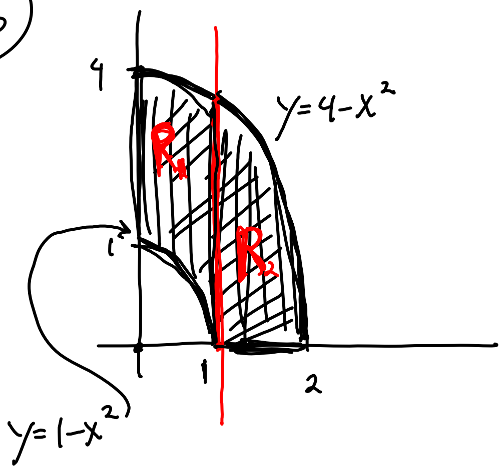
$$\int_0^1 \int_{2x}^{3x} x+y^2 \, dy\, dx = \int_0^1 xy + \frac{1}{3}y^3 \Big]_{y=2x}^{y=3x} dx = \int_0^1 \left(x(3x) - x(2x)\right) + \left(\frac{1}{3}(3x)^3 - \frac{1}{3}(2x)^3\right) dx$$

$$= \int_0^1 x^2 + \frac{1}{3}\left(27x^3 - 8x^3\right) dx = \int_0^1 x^2 + \frac{19}{3}x^3 \, dx = \frac{1}{3}x^3 + \frac{19}{12}x^4 \Big]_{x=0}^{x=1}$$

$$= \frac{1}{3} + \frac{19}{12} = \frac{23}{12}$$

82 omit

86



$$R_2 = \boxed{\begin{array}{c} 1 \le x \le 2 \\ 0 \le y \le 4-x^2 \end{array}}$$

$$R_1 = \boxed{\begin{array}{c} 0 \le x \le 1 \\ 1-x^2 \le y \le 4-x^2 \end{array}}$$

$$\iint_R x\,dA = \iint_{R_1} x\,dA + \iint_{R_2} x\,dA$$

$$= \int_0^1 \int_{1-x^2}^{4-x^2} x\,dy\,dx + \int_1^2 \int_0^{4-x^2} x\,dy\,dx$$

$$= \int_0^1 x\left(4-x^2-(1-x^2)\right)dx + \int_1^2 x\left(4-x^2-0\right)dx$$

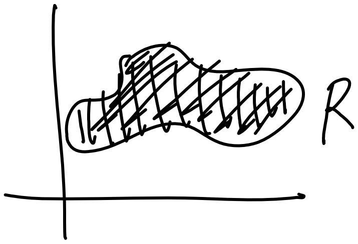
$$= \int_0^1 3x\,dx + \int_1^2 4x-x^3\,dx$$

$$= \frac{3}{2}x^2\Big]_{x=0}^{x=1} + 2x^2-\frac{1}{4}x^4\Big]_{x=1}^{x=2}$$

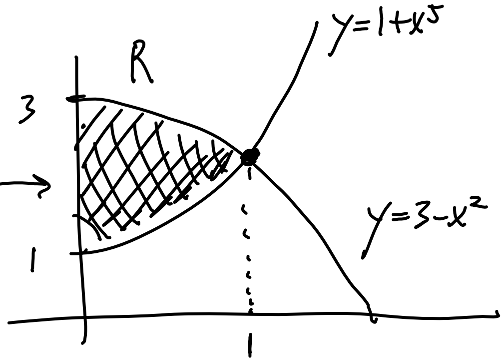
$$= \frac{3}{2} + (8-2) - \left(4-\frac{1}{4}\right)$$

$$= \frac{3}{2} + 6 - \frac{15}{4} = \frac{6+24-15}{4} = \frac{15}{4}$$

(90)



$$\text{Area}(R) = \iint_R 1 \, dA$$



$$\boxed{\begin{array}{l} 0 \le x \le 1 \\ 1+x^5 \le y \le 3-x^2 \end{array}}$$

$$3-x^2 = 1+x^5$$

$$0 = x^5 + x^2 - 2$$

note That $x=1$ yields a solution

$$\text{Area} = \iint_R 1 \, dA = \int_0^1 \int_{1+x^5}^{3-x^2} 1 \, dy \, dx$$

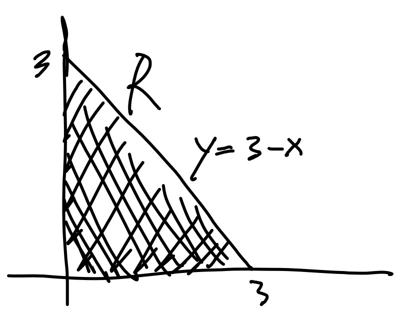
$$= \int_0^1 3-x^2-(1+x^5) \, dx$$

$$= \int_0^1 2-x^2-x^5 \, dx$$

$$= 2x - \tfrac{1}{3}x^3 - \tfrac{1}{6}x^6 \Big]_{x=0}^{x=1}$$

$$= 2 - \tfrac{1}{3} - \tfrac{1}{6} = 2 - \tfrac{1}{2} = \frac{3}{2}$$

(94)



Average value of $f(x,y) = \sin(y)$ over This Region

Type 1 $\boxed{\begin{array}{l} 0 \le x \le 3 \\ 0 \le y \le 3-x \end{array}}$

$$\frac{1}{\text{Area}(R)} \iint_R f(x,y) \, dA = \frac{2}{9} \int_0^3 \int_0^{3-x} \sin(y) \, dy \, dx$$

$$= \frac{2}{9}\int_0^3 -\cos(y)\Big]_{y=0}^{y=3-x} dx = \frac{2}{9}\int_0^3 1-\cos(3-x)\,dx = \frac{2}{9}\left(x+\sin(3-x)\Big]_{x=0}^{x=3}\right)$$

let $u=3-x$

$$= \frac{2}{9}\Big(3+0-\big(0+\sin(3)\big)\Big) = \boxed{\frac{2}{9}\big(3-\sin(3)\big)}$$

96

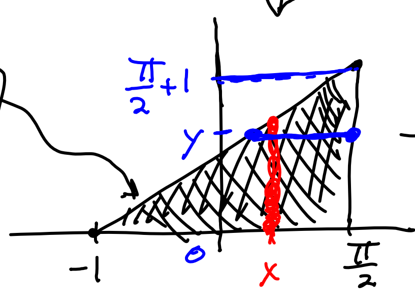
$$\int_{-1}^{\frac{\pi}{2}}\int_0^{x+1} \sin(x)\,dy\,dx$$

Reverse the order of integration

$$R = \boxed{\begin{matrix} -1 \le x \le \frac{\pi}{2} \\ 0 \le y \le x+1 \end{matrix}} \text{ Type 1}$$

$y = x+1$

$x = y-1$



Type 2

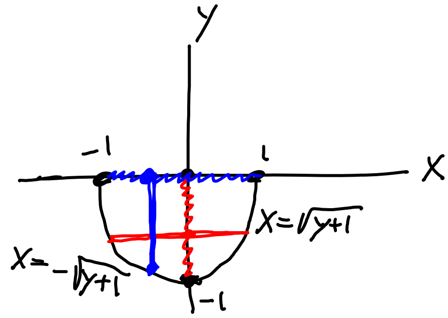
$$\boxed{\begin{matrix} 0 \le y \le \frac{\pi+2}{2} \\ y-1 \le x \le \frac{\pi}{2} \end{matrix}}$$

$$\boxed{\int_{-1}^{\frac{\pi}{2}}\int_0^{x+1} \sin(x)\,dy\,dx = \int_0^{\frac{\pi+2}{2}}\int_{y-1}^{\frac{\pi}{2}} \sin(x)\,dx\,dy}$$ → evaluate on your own

(98) $$\int_{-1}^{0}\int_{-\sqrt{y+1}}^{\sqrt{y+1}} y^2\,dx\,dy$$

Type 2

$$R = \boxed{\begin{array}{c} -1 \le y \le 0 \\ -\sqrt{y+1} \le x \le \sqrt{y+1} \end{array}} \longrightarrow$$



$x = \pm\sqrt{y+1}$

$x^2 = y+1$

$y = x^2-1$

Type 1

$$\boxed{\begin{array}{c} -1 \le x \le 1 \\ x^2-1 \le y \quad 0 \end{array}}$$

Therefore

$$\boxed{\int_{-1}^{0}\int_{-\sqrt{y+1}}^{\sqrt{y+1}} y^2\,dx\,dy = \int_{-1}^{1}\int_{x^2-1}^{0} y^2\,dy\,dx}$$ evaluate on your own.

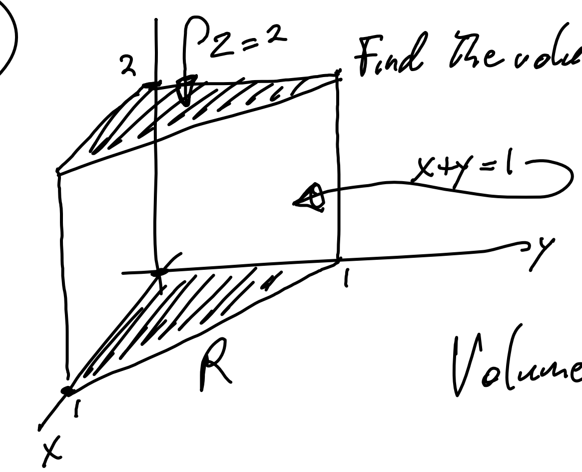
$\frac{32}{105}$

Give 97 + 99 a try on your own.

On Exam 3 I'll ask to rewrite a type 1 as type 2 or type 2 as type 1 without evaluating.

102 on your own

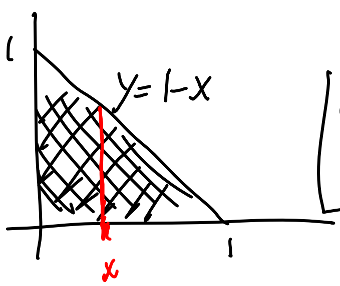
108



Find the volume

$$\text{Volume} = \iint_R 2\,dA \quad \text{which is clearly} = \tfrac{1}{2}(2) = 1$$

but let's calculate the integral anyway.



$$\boxed{\begin{aligned} 0 \le x \le 1 \\ 0 \le y \le 1-x \end{aligned}}$$

$$\iint_R 2\,dA = \int_0^1\int_0^{1-x} 2\,dy\,dx = \int_0^1 2(1-x)\,dx$$

$$= \int_0^1 2-2x\,dx = 2x - x^2\Big]_0^1 = 2-1-0 = 1 \checkmark$$