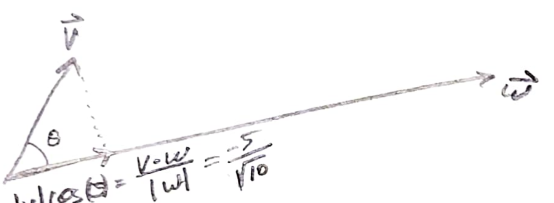
Math2320 – Fall 2022 – Exam 1 Name: Solutions

(10 pts) (1) Find the equation of the sphere centered at $(1, 0, 1)$ and containing the point $(3, 4, -1)$ on its surface.

$$r = \sqrt{(-2)^2 + (-4)^2 + (2)^2} = \sqrt{4 + 16 + 4} = \sqrt{24}$$

$$\boxed{(x-1)^2 + y^2 + (z-1)^2 = 24}$$

(14 pts) (2) (a) Find the projection of the vector $\vec{v} = \langle 1, 1, -2 \rangle$ onto the vector $\vec{w} = \langle 1, 0, 3 \rangle$.



$$\text{proj}_w(\vec{v}) = \frac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \vec{w} = \frac{1+0-6}{1+0+9} \langle 1, 0, 3 \rangle$$

$$= -\frac{1}{2} \langle 1, 0, 3 \rangle = \boxed{\left\langle \frac{-1}{2}, 0, \frac{-3}{2} \right\rangle}$$

(b) Approximate the angle (with two decimal-place accuracy) between the vectors $\vec{v}$ and $\vec{w}$.

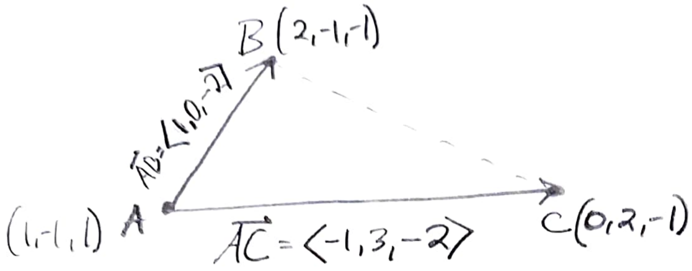
$$\cos(\theta) = \frac{\vec{v} \cdot \vec{w}}{|\vec{v}||\vec{w}|} = \frac{-5}{\sqrt{6}\sqrt{10}}$$

$$\theta = \text{Arccos}\left(\frac{-5}{\sqrt{60}}\right) \approx \boxed{130.20^\circ \approx 2.27 \text{ radians}}$$

16 pts

(3) Consider the points $A(1,-1,1)$, $B(2,-1,-1)$, and $C(0,2,-1)$.

(a) Find the area of the triangle ABC.



$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -2 \\ -1 & 3 & -2 \end{vmatrix} = \langle 6, 4, 3 \rangle$$

$$|\vec{AB} \times \vec{AC}| = \sqrt{36+16+9} = \sqrt{61}$$

$$\boxed{\text{Area} = \tfrac{1}{2}\sqrt{61}}$$

(b) Find the Cartesian equation of the plane containing the triangle ABC.

$\vec{n} \cdot \langle x\ y\ z \rangle = \vec{n} \cdot \langle 0, 2, -1 \rangle$ with $\vec{n} = \vec{AB} \times \vec{AC}$

$$\boxed{6x + 4y + 3z = 5}$$

(4) Consider the two lines given by the following sets of parametric equations.

(12 pts)

$$L_1 \quad x = 3 + 2t \qquad\qquad L_2 \quad x = 1 + t$$
$$y = 1 - t \qquad \text{and} \qquad y = 1 + t$$
$$z = t \qquad\qquad\qquad z = 2 - t$$

(a) Are these lines orthogonal, parallel, or neither?

Slope L_1: $\vec{V}_1 = \langle 2, -1, 1 \rangle$

Slope L_2: $\vec{V}_2 = \langle 1, 1, -1 \rangle$

not scalar multiples and $\vec{v}_1 \cdot \vec{v}_2 = 0$

~~neither parallel nor~~ $\boxed{\text{orthogonal}}$

(b) Do these lines intersect at a point?

If they do, then there is a solution to the following

$$\boxed{\begin{aligned} 3+2t &= 1+s \\ 1-t &= 1+s \\ t &= 2-s \end{aligned}} \quad \begin{aligned} &\longrightarrow 2t - s = -2 \\ &\longrightarrow \underline{-(-t-s = 0)} \\ &3t = -2 \\ &\boxed{t = \tfrac{-2}{3}} \\ &\boxed{s = \tfrac{2}{3}} \end{aligned}$$

$$-\frac{2}{3} = 2 - \frac{2}{3}$$

$$-\frac{2}{3} \neq \frac{4}{3}$$

So no point of intersection.

(10 pts) (5) Given the point $(4, \frac{\pi}{6}, \frac{\pi}{3})$ in spherical coordinates, find the Cartesian coordinates (x, y, z) with 2-decimal-place accuracy.

$$X = \rho \cos(\theta) \sin(\varphi) = 4 \tfrac{\sqrt{3}}{2} \tfrac{\sqrt{3}}{2} = 3$$

$$y = 4 \sin(\theta) \sin(\varphi) = 4 \tfrac{1}{2} \tfrac{\sqrt{3}}{2} = \sqrt{3}$$

$$z = 4 \cos(\varphi) = 4 \tfrac{1}{2} = 2$$

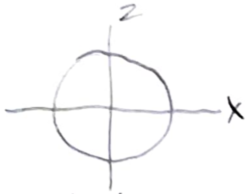
$$\boxed{(3, \sqrt{3}, 2)}$$

12 pts

(6) Sketch the surface given by the equation $x^2 + z^2 = y^2$.

$y = k$

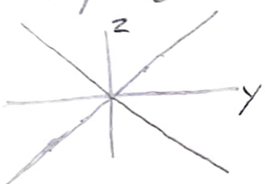
$x^2 + z^2 = k^2$

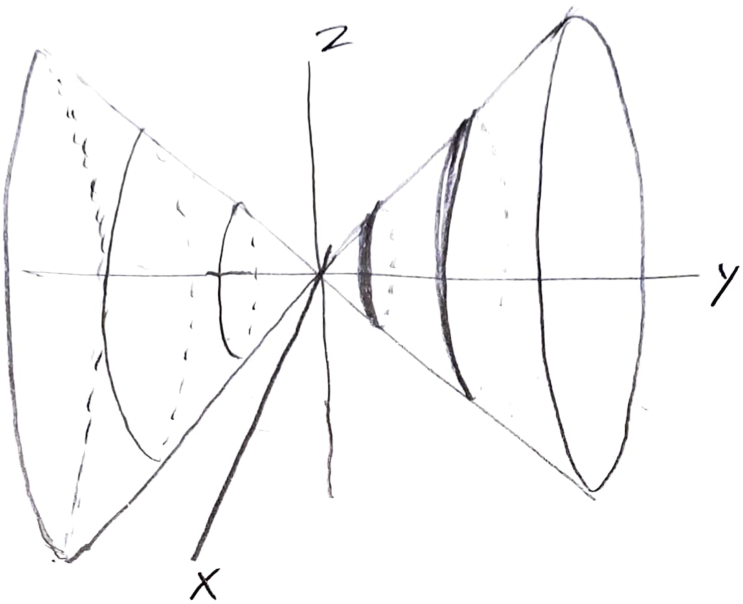


circles parallel to xz-plane

$x = 0$

$z^2 - y^2 = 0$





(7) Find a vector function $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ which traces out the intersection of the two cylinders $y^2 + z^2 = 1$ and $x^2 + z^2 = 1$.

10 pts

vector function must satisfy $x^2 + z^2 = 1$ and $y^2 + z^2 = 1$

$$\boxed{\vec{r}(t) = \langle \cos(t), \cos(t), \sin(t) \rangle \quad 0 \le t \le 2\pi}$$

(8) The acceleration function of a particle traveling along a curve in 2 dimensions is given by $\vec{a}(t) = \langle 2t, \sin(t) \rangle$.

16 pts

(a) Given that $\vec{v}(0) = \langle 1, -1 \rangle$, find the velocity function $\vec{v}(t)$.

(b) Given that $\vec{r}(0) = \langle 0, 0 \rangle$, find the position function $\vec{r}(t)$.

(c) Find the speed function.

$$\vec{a}(t) = \langle 2t, \sin(t) \rangle$$

$$\vec{V}(t) = \langle t^2, -\cos(t) \rangle + \vec{C}$$

$$\langle 1, -1 \rangle = \langle 0, -1 \rangle + \vec{C}$$

$$\langle 1, 0 \rangle = \vec{C}$$

$$\boxed{\vec{V}(t) = \langle t^2+1, -\cos(t) \rangle}$$

$$\vec{r}(t) = \langle \tfrac{1}{3}t^3+t, -\sin(t) \rangle + \vec{C}$$

$$\langle 0, 0 \rangle = \langle 0, 0 \rangle + C$$

$$\boxed{\vec{r}(t) = \langle \tfrac{1}{3}t^3, -\sin(t) \rangle}$$

$$\boxed{\text{speed} = |\vec{V}(t)| = \sqrt{t^4+2t^2+1+\cos^2(t)}}$$