Section 3.4 Antiderivatives of rational functions using partial-fraction decompositions.

A **rational function** is a ratio of two polynomials.

Some easy examples of rational functions and their antiderivatives are as follows.

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\int \underbrace{\frac{a}{ax+b}}_{\frac{g'(x)}{g(x)}}\,dx = \frac{1}{a}\ln|ax+b| + C$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\,\mathrm{Arctan}\left(\frac{x}{a}\right) + C$$

$$\int \frac{x}{a^2+x^2}\,dx = \frac{1}{2}\int \underbrace{\frac{2x}{a^2+x^2}}_{\frac{g'(x)}{g(x)}}\,dx = \frac{1}{2}\ln(a^2+x^2) + C$$

Lots of other rational functions decompose into these basic forms.

$$\frac{p(x)}{x(x-1)^3(x+2)^2(x^2+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3} + \frac{E}{x+2} + \frac{F}{(x+2)^2} + \frac{Gx+H}{x^2+1}$$

This decomposition will work as long as degree of the numerator is strictly less than the degree of the denominator.

<u>example</u>

$$\int \frac{4x+4}{x^2(x+2)}\,dx$$

$$\frac{4x+4}{x^2(x+2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2}$$ Now solve for A, B, C

$$\boxed{\frac{4x+4}{x^2(x+2)}} = \frac{Ax(x+2)}{x^2(x+2)} + \frac{B(x+2)}{x^2(x+2)} + \frac{Cx^2}{x^2(x+2)} = \boxed{\frac{Ax(x+2)+B(x+2)+Cx^2}{x^2(x+2)}}$$

$$\underline{4x+4 = Ax(x+2) + B(x+2) + Cx^2}$$

<u>Method 1</u> Pick values of x to get different equations for A, B, C

<u>$x=-2$</u>

$-4 = 0+0+4C$

$\boxed{-1=C}$

<u>$x=0$</u>

$4 = 0+2B+0$

$\boxed{2=B}$

<u>$x=-1$</u>

$0 = -A+B+C$

$0 = -A+2-1$

$-1 = -A$

$\boxed{1=A}$

$$4x+4 = Ax(x+2) + B(x+2) + Cx^2$$

Method 2 Collect like powers of x and compare coefficients.

$$4x+4 = (A+C)x^2 + (2A+B)x + 2B$$

so now

$$\boxed{\begin{aligned} &(1)\quad 0 = A+C \\ &(2)\quad 4 = 2A+B \\ &(3)\quad 4 = 2B \end{aligned}}$$

(3) yields $\boxed{B=2}$

(2) yields $4 = 2A+2$
$\boxed{1=A}$

(1) yields $0 = 1+C$
$\boxed{-1=C}$

$$\int \frac{4x+4}{x^2(x+2)}\,dx = \int \frac{1}{x} + \frac{2}{x^2} - \frac{1}{x+2}\,dx = \int \frac{1}{x} + 2x^{-2} - \frac{1}{x+2}\,dx$$

$$\boxed{= \ln|x| - \frac{2}{x} - \ln|x+2| + C}$$

example

$$\int \frac{3x^2+x+1}{x(x^2+1)}\,dx$$

$$\frac{3x^2+x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$\frac{3x^2+x+1}{x(x^2+1)} = \frac{A(x^2+1)+(Bx+C)x}{x(x^2+1)}$$

$$3x^2+x+1 = A(x^2+1)+(Bx+C)x$$

$x=0$

$1 = A + 0$

$\boxed{1=A}$

$x=1$

$5 = 2A + B + C$

$\boxed{3 = B+C}$

$x=-1$

$3 = 2A + (-B+C)(-1)$

$3 = 2 + B - C$

$\boxed{1 = B - C}$

$$\begin{array}{rl} & 3 = B+C \\ + & 1 = B-C \\ \hline & 4 = 2B \end{array}$$

$2 = B$

$3 = 2 + C$

$\boxed{1 = C}$

$$\int \frac{3x^2+x+1}{x(x^2+1)}\,dx = \int \frac{1}{x} + \frac{2x+1}{x^2+1}\,dx = \int \frac{1}{x} + \underbrace{\frac{2x}{x^2+1}}_{\frac{g'(x)}{g(x)}} + \frac{1}{x^2+1}\,dx$$

$$\boxed{= \ln|x| + \ln(x^2+1) + \operatorname{Arctan}(x) + C}$$

What if degree(numerator) $\geq$ degree(denominator) ?

First apply long division as follows.

<u>example</u> $\displaystyle\int \frac{r^3+1}{r(r+4)}\,dr$

$$\begin{array}{r l}
 & r-4 \quad \text{quotient} \\
r^2+4r \, \big) & r^3 + 1 \\
 & -(r^3+4r^2) \\
\hline
 & -4r^2+1 \\
 & -(-4r^2-16r) \\
\hline
 & 16r+1 \quad \text{degree } 1 < \text{deg of division.} \\
 & \text{remainder}
\end{array}$$

So new $\displaystyle \frac{r^3+1}{r(r+4)} = r-4 + \frac{16r+1}{r(r+4)}$

So $$\int \frac{r^3+1}{r(r+4)}\,dr = \int \underline{r-4} + \frac{16r+1}{r(r+4)}\,dr$$

just a polynomial

use partial fraction decomposition

$$\frac{16r+1}{r(r+4)} = \frac{A}{r} + \frac{B}{r+4}$$

$$16r+1 = A(r+4) + Br$$

$\underline{r=0}$	$\underline{r=-4}$
$1 = 4A$	$-63 = -4B$
$\frac{1}{4} = A$	$\frac{63}{4} = B$

$$\int \frac{r^3+1}{r(r+4)}\,dr = \int r-4 + \frac{1}{4}\frac{1}{r} + \frac{63}{4}\frac{1}{r+4}\,dr$$

$$\boxed{= \frac{1}{2}r^2 - 4r + \frac{1}{4}\ln|r| + \frac{63}{4}\ln|r+4| + C}$$