Section 3.3 Trigonometric substitutions.

expression	substitution	useful identity
$a^2 - x^2$	$x = a\sin(\theta)$	$\sin^2(\theta) + \cos^2(\theta) = 1$
$x^2 - a^2$	$x = a\sec(\theta)$	$\sec^2(\theta) - 1 = \tan^2(\theta)$
$a^2 + x^2$	$x = a\tan(\theta)$	$\tan^2(\theta) + 1 = \sec^2(\theta)$

example

$$\int \frac{x^3}{\sqrt{4-x^2}}\,dx = \int \frac{8\sin^3(\theta)}{\sqrt{4-4\sin^2(\theta)}}\,2\cos(\theta)\,d\theta = 16\int \frac{\sin^3(\theta)}{2\sqrt{1-\sin^2(\theta)}}\cos(\theta)\,d\theta$$

$x = 2\sin(\theta)$

$dx = 2\cos(\theta)\,d\theta$

$$= 8\int \frac{\sin^3(\theta)}{\sqrt{\cos^2(\theta)}}\cos(\theta)\,d\theta = 8\int \frac{\sin^3(\theta)}{|\cos(\theta)|}\cos(\theta)\,d\theta =$$

assume $\cos(\theta) \geq 0$ on the interval of integration.

$$= 8\int \frac{\sin^3(\theta)}{\cancel{\cos(\theta)}}\cancel{\cos(\theta)}\,d\theta = 8\int \sin^3(\theta)\,d\theta = 8\int \sin^2(\theta)\sin(\theta)\,d\theta$$

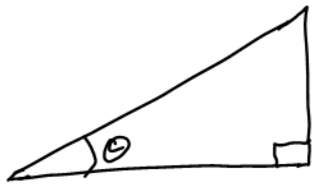
$$= 8\int (1-\cos^2(\theta))\sin(\theta)\,d\theta = -8\int (1-u^2)\,du = \frac{8}{3}u^3 - 8u + C$$

let $u = \cos(\theta)$

$du = -\sin(\theta)\,d\theta$

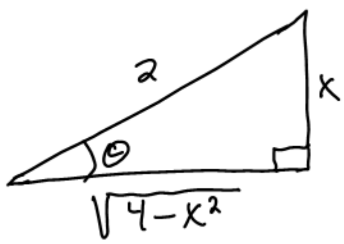
$$= \frac{8}{3}\cos^3(\theta) - 8\cos(\theta) + C = \frac{8}{3}\left(\frac{\sqrt{4-x^2}}{2}\right)^3 - 8\,\frac{\sqrt{4-x^2}}{2} + C$$

$$\boxed{= \frac{1}{3}(4-x^2)^{\frac{3}{2}} - 4\sqrt{4-x^2} + C}$$



original substitution $\rightarrow$ $x = 2\sin(\theta)$

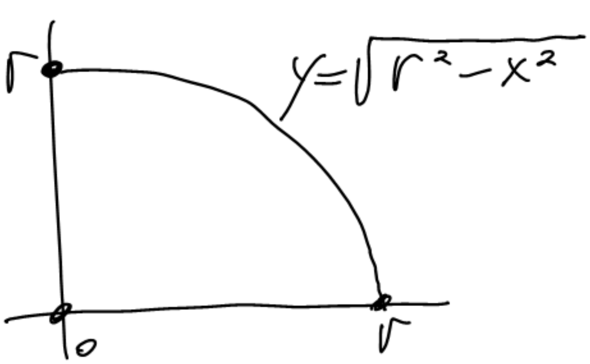
$$\frac{x}{2} = \sin(\theta)$$



$\rightarrow \cos(\theta) = \dfrac{\sqrt{4-x^2}}{2}$

<u>example</u> The area of a circle πr^2.

The $\frac{1}{4}$ circle of radius r is the region under the curve $y = \sqrt{r^2 - x^2}$ $\quad 0 \le x \le r$.



so Area of a circle $= 4\displaystyle\int_0^r \sqrt{r^2 - x^2}\,dx$

let $x = r\sin(\theta)$

$dx = r\cos(\theta)\,d\theta$

$$= 4\int_{x=0}^{x=r} \sqrt{r^2 - r^2\sin^2(\theta)}\, r\cos(\theta)\, d\theta = 4\int_{\theta=0}^{\theta=\frac{\pi}{2}} r\sqrt{1-\sin^2(\theta)}\, r\cos(\theta)\, d\theta$$

$$= 4r^2\int_0^{\frac{\pi}{2}} \sqrt{\cos^2(\theta)}\, \cos(\theta)\, d\theta = 4r^2\int_0^{\frac{\pi}{2}} |\cos(\theta)|\cos(\theta)\, d\theta =$$

$\cos(\theta) \geq 0$ for $0 \leq \theta \leq \frac{\pi}{2}$

$$= 4r^2\int_0^{\frac{\pi}{2}} \cos^2(\theta)\, d\theta = 4r^2\int_0^{\frac{\pi}{2}} \frac{1}{2}\left(1+\cos(2\theta)\right) d\theta$$

$$= 2r^2\int_0^{\frac{\pi}{2}} 1+\cos(2\theta)\, d\theta = 2r^2\left(\theta + \frac{1}{2}\sin(2\theta)\right)\Bigg]_{\theta=0}^{\theta=\frac{\pi}{2}}$$

$$= 2r^2\left(\frac{\pi}{2} + 0 - (0+0)\right) = \pi r^2$$

example

$$\int \frac{1}{(1+x^2)^2}\,dx = \int \frac{1}{(1+\tan^2(\theta))^2}\sec^2(\theta)\,d\theta = \int \frac{1}{(\sec^2(\theta))^2}\sec^2(\theta)\,d\theta$$

let $x = \tan(\theta)$

$dx = \sec^2(\theta)\,d\theta$

$$= \int \frac{1}{\sec^2(\theta)}\,d\theta = \int \cos^2(\theta)\,d\theta = \int \frac{1}{2}(1+\cos(2\theta))\,d\theta$$

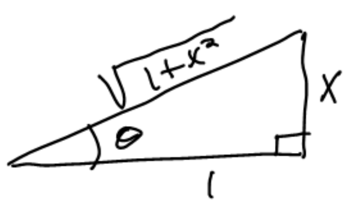
$$= \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta) + C = \frac{1}{2}\theta + \frac{1}{2}\sin(\theta)\cos(\theta) + C$$

original substitution

$x = \tan(\theta)$

$\frac{x}{1} = \tan(\theta)$

$$\boxed{= \frac{1}{2}\text{Arctan}(x) + \frac{1}{2}\,\frac{x}{1+x^2} + C}$$



$\sin(\theta) = \frac{x}{\sqrt{1+x^2}}$

$\cos(\theta) = \frac{1}{\sqrt{1+x^2}}$

$\theta = \text{Arctan}(x)$