3.1 Integration by "parts"

Recall the product rule

$$\frac{d}{dx}u(x)v(x) = u'(x)v(x) + u(x)v'(x)$$

Take the anti-derivative of both sides.

$$\int \frac{d}{dx}u(x)v(x)\,dx = \int u'(x)v(x) + u(x)v'(x)\,dx$$

$$u(x)v(x) = \int v(x)u'(x)\,dx + \int u(x)v'(x)\,dx$$

using some shorthand notation $u'(x)\,dx = du$ and
$v'(x)\,dx = dv$

This now becomes

$$uv = \int v\,du + \int u\,dv$$

Rewrite as

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Integration by parts formula.

example

$$\int x\ln(x)\,dx = \frac{1}{2}x^2\ln(x) - \int \frac{1}{2}x^2\frac{1}{x}\,dx = \frac{1}{2}x^2\ln(x) - \frac{1}{2}\int x\,dx$$

$$\boxed{= \frac{1}{2}x^2\ln(x) - \frac{1}{4}x^2 + C}$$

let $u = \ln(x)$ $\quad du = \frac{1}{x}dx$

$dv = x\,dx \quad v = \frac{1}{2}x^2$

Check

$$\frac{d}{dx}\left(\frac{1}{2}x^2\ln(x) - \frac{1}{4}x^2\right) = \frac{1}{2}\left(2x\ln(x) + \frac{1}{\cancel{2}}\cancel{x^2}\frac{1}{x} - \frac{1}{\cancel{2}}x\right) = x\ln(x). \checkmark$$

example

$$\int_a^b u\,dv = uv\Big]_a^b - \int_a^b v\,du$$

or calculate $\int u\,dv$ and evaluate afterwards.

$$\int_0^1 x^2 e^{-x}\,dx$$

$$\int x^2 e^{-x}\,dx = -x^2e^{-x} - \int 2x(-e^{-x})\,dx = -x^2e^{-x} + 2\int xe^{-x}\,dx$$

let $u = x^2$ $\quad du = 2x\,dx$

$dv = e^{-x}dx \quad v = -e^{-x}$

use another application of integration by parts

$u = x \quad du = dx$

$dv = e^{-x}dx \quad v = -e^{-x}$

$$= -x^2e^{-x} + 2\left[-xe^{-x} - \int -e^{-x}\,dx\right] = -x^2e^{-x} - 2xe^{-x} + 2\int e^{-x}\,dx$$

$$= -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C = \boxed{-e^{-x}(x^2+2x+2) + C}$$

$$\int_0^1 x^2e^{-x}\,dx = -e^{-x}(x^2+2x+2)\Big]_0^1 = -5e+2 = \boxed{2-5e}$$

Often choosing u first and dv second in The following order will work.

Logs
Inverse trig
Polynomials
Exponentials
Trig functions.

example

$$\int e^{2x}\cos(x)\,dx = e^{2x}\sin(x) - 2\int e^{2x}\sin(x)\,dx =$$

$$\boxed{\begin{array}{ll} u = e^{2x} & du = 2e^{2x}dx \\ dv = \cos(x)\,dx & v = \sin(x) \end{array}}$$

$u = e^{2x}$ $du = 2e^{2x}dx$

$dv = \sin(x)\,dv$ $v = -\cos(x)\,dx$

$$= e^{2x}\sin(x) - 2\left[-2e^{2x}\cos(x) - \int -2e^{2x}\cos(x)\,dx\right]$$

Rewrite The whole equation

$$\int e^{2x}\cos(x)\,dx = e^{2x}\sin(x) + 4e^{2x}\cos(x) - 4\int e^{2x}\cos(x)\,dx$$

$+4\int e^{2x}\cos(x)\,dx$ $\qquad$ $+4\int e^{2x}\cos(x)\,dx$

$$5\int e^{2x}\cos(x)\,dx = e^{2x}\sin(x) + 4e^{2x}\cos(x)$$

$$\boxed{\int e^{2x}\cos(x)\,dx = \frac{1}{5}e^{2x}\left(\sin(x)+4\cos(x)\right) + C}$$