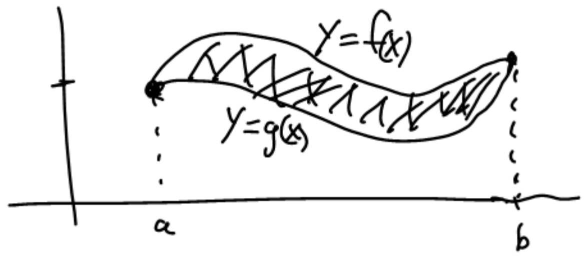
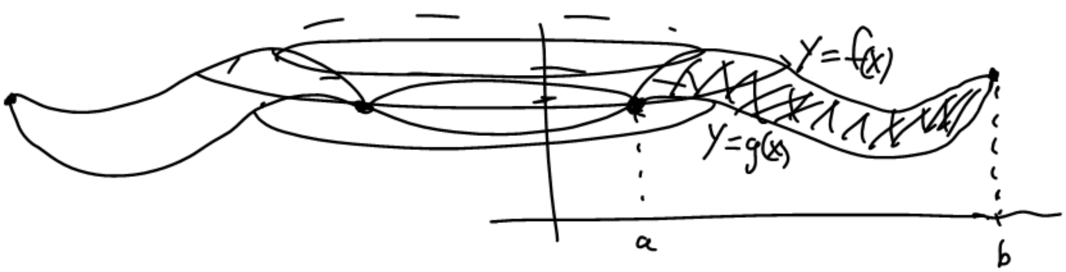
2.3 Volumes using cylindrical shells

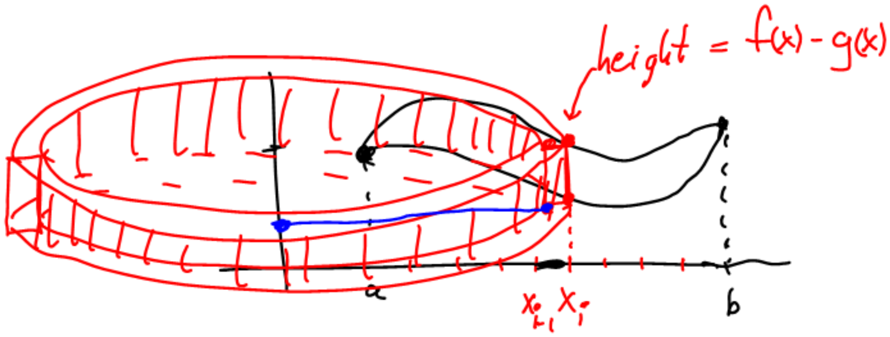
Consider a region as shown



Rotate around the y-axis and calculate the resulting volume.



Subdivide $a \le x \le b$ into subintervals of length $\Delta x = \frac{b-a}{n}$ and let $x_i = a + i\Delta x$



Cylindrical shell at x_i

height = $f(x) - g(x)$

width = Δx

inner radius = x_{i-1}

outer radius = x_i

Volume of this shell at x_i is

$$\underbrace{\pi x_i^2 (f(x_i)-g(x_i))}_{\text{volume of outer cylinder}} - \underbrace{\pi x_{i-1}^2 (f(x_i)-g(x_i))}_{\text{volume of the inner cylinder}}$$

$$= \pi (f(x_i)-g(x_i))(x_i^2 - x_{i-1}^2)$$

$$= \pi (f(x_i)-g(x_i))(x_i + x_{i-1})\underbrace{(x_i - x_{i-1})}$$

$$= \pi (f(x_i)-g(x_i))(x_i + x_{i-1}) \Delta x$$

$$\approx \pi (f(x_i)-g(x_i)) 2x_i \Delta x$$

when Δx is really tiny $x_i + x_{i-1} \approx 2x_i$

The total volume of all n shells is

$$\approx \sum_{i=1}^{n} 2\pi x_i (f(x_i)-g(x_i)) \Delta x$$

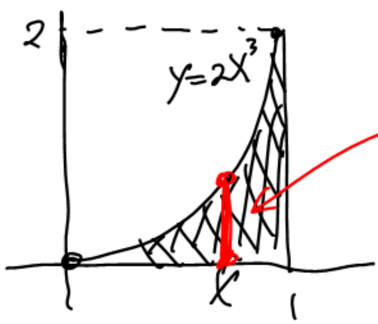
The estimate of the volume with these shells gets better as $n \to \infty$. Therefore

$$\text{Volume} = \lim_{n \to \infty} \sum_{i=1}^{n} 2\pi x_i (f(x_i)-g(x_i)) \Delta x$$

$$\boxed{\text{So} \quad \text{Volume} = \int_a^b 2\pi x\,(f(x)-g(x))\,dx \quad \text{by definition of integral}}$$

example $\boxed{\#83 \text{ section } 2.2}$

We calculated the volume of the following region when rotated around the y-axis using annulus-shaped slices.

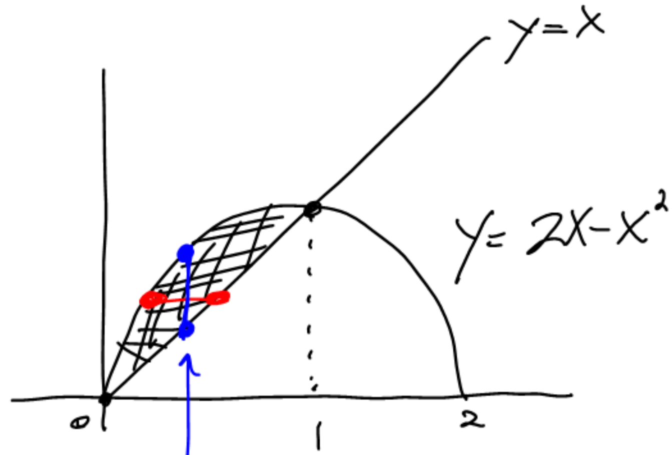


height of the shells is $2x^3-0=2x^3$

$$\text{Volume} = \int_0^1 2\pi x\, 2x^3\,dx$$

$$= \int_0^1 4\pi x^4\,dx = \frac{4}{5}\pi x^5\Big]_0^1 = \boxed{\frac{4}{5}\pi}$$

example Rotate the region shown around the y-axis and calculate the volume

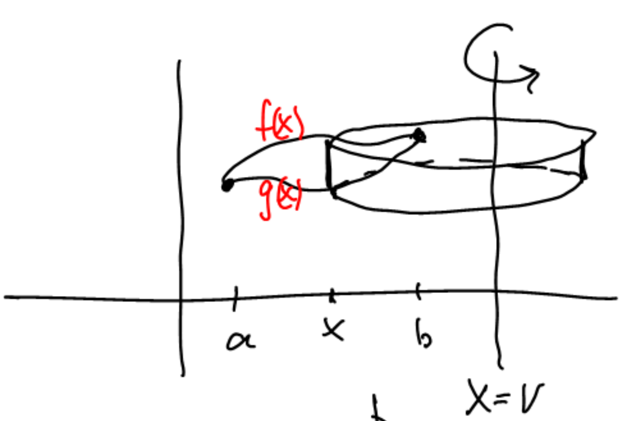


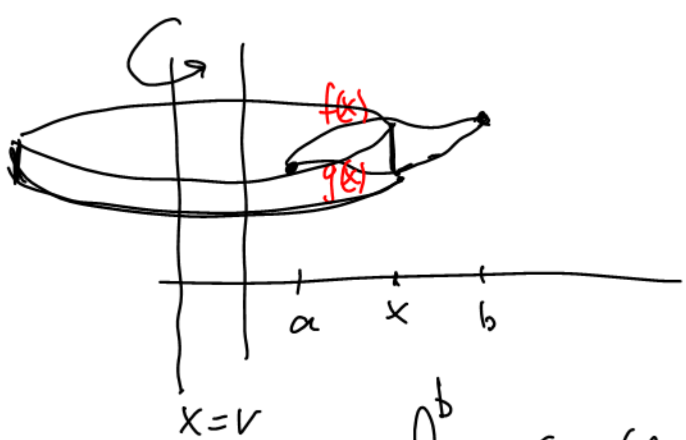
height of shells $= f(x) - g(x) = (2x - x^2) - (x) = \boxed{x - x^2}$

$$\text{Volume} = \int_0^1 2\pi x(x - x^2)\,dx = 2\pi \int_0^1 x^2 - x^3\,dx = 2\pi\left(\tfrac{1}{3}x^3 - \tfrac{1}{4}x^4\right)\Big]_0^1$$

$$= 2\pi\left(\tfrac{1}{3} - \tfrac{1}{4}\right) = \frac{\pi}{6}$$

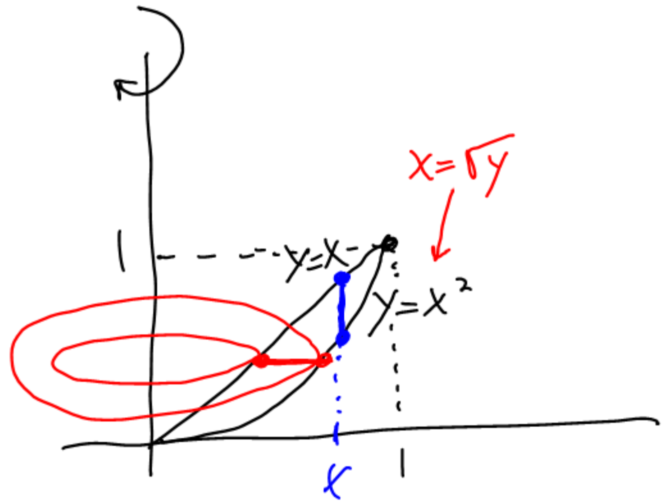
Using a different vertical axis.





$$\text{Volume} = \int_a^b 2\pi(v-x)(f(x)-g(x))\,dx \qquad \text{Volume} = \int_a^b 2\pi(x-v)(f(x)-g(x))\,dx$$

Here is a region which will rotate around the y-axis. Let's calculate the volume using both shells and slices.



Slices

$$A(x) = \pi R^2 - \pi r^2$$
$$= \pi(\sqrt{y})^2 - \pi y^2$$
$$= \pi(y - y^2)$$

$$\text{Volume} = \int_0^1 A(x)\,dx = \int_0^1 \pi(y-y^2)\,dy$$
$$= \pi\left(\tfrac{1}{2}y^2 - \tfrac{1}{3}y^3\right)\Big]_0^1 = \pi\left(\tfrac{1}{2}-\tfrac{1}{3}\right)$$
$$= \frac{\pi}{6}$$

Shells

height of shell at x is $(x - x^2)$

$$\text{Volume} = \int_0^1 2\pi x(\text{height})\,dx$$
$$= \int_0^1 2\pi x(x - x^2)\,dx$$
$$= 2\pi\int_0^1 x^2 - x^3\,dx = 2\pi\left(\tfrac{1}{3}x^3 - \tfrac{1}{4}x^4\right)\Big]_{x=0}^{x=1} = 2\pi\left(\tfrac{1}{3}-\tfrac{1}{4}\right) = \frac{2\pi}{12}$$
$$= \frac{\pi}{6}$$

Same.