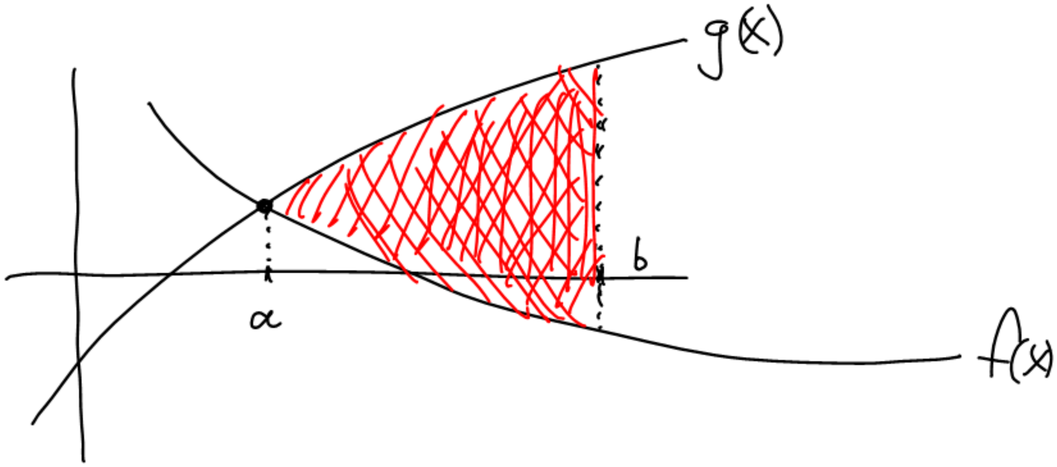
Section 2.1

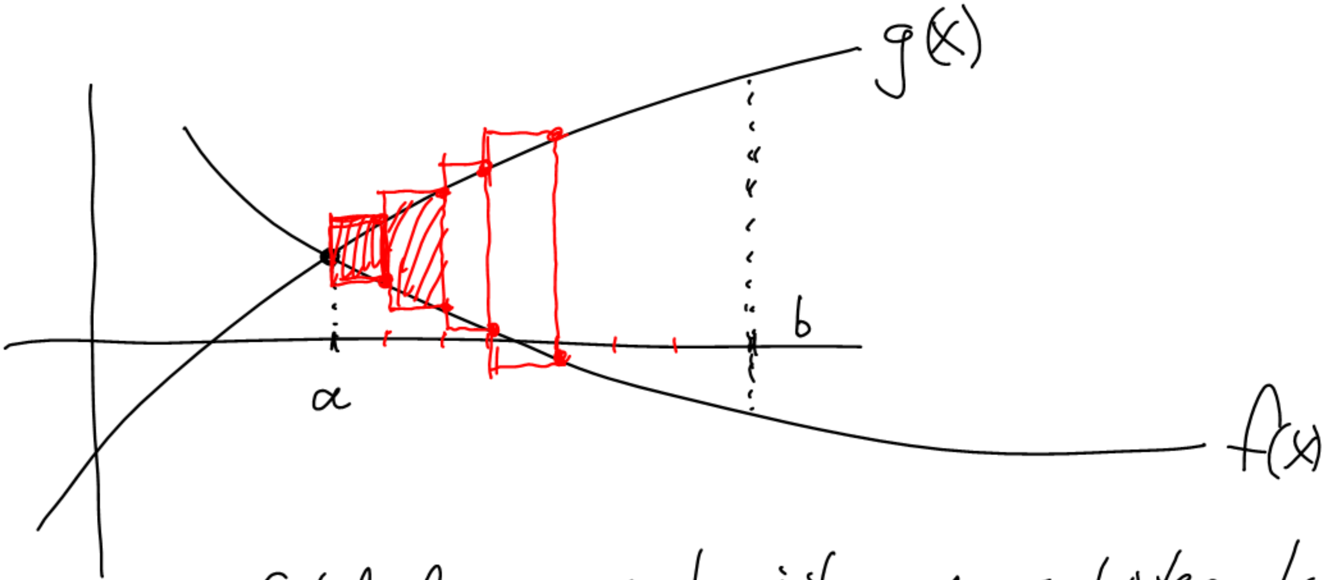
More on Areas.

Suppose we have two continuous functions

$f(x) \le g(x)$ on interval $a \le x \le b$.



How can we calculate the area between $f(x)$ and $g(x)$ on $a \le x \le b$??



Subdivide $a \le x \le b$ into n subintervals of length $\Delta x = \frac{b-a}{n}$

let $x_i = a + i\Delta x$

at each x_i consider the rectangle whose height is $g(x_i) - f(x_i)$ and whose width is Δx.

The area of this rectangle is $\left(g(x_i) - f(x_i)\right)\Delta x$

The sum

$$\sum_{i=1}^{n} \left(g(x_i) - f(x_i)\right)\Delta x$$

A Riemann Sum

Estimates the area we want to measure.

The more rectangles there are the better the estimate gets.

Therefore

$$\text{Area between } f(x) \text{ and } g(x) = \lim_{n\to\infty} \sum_{i=1}^{n} \left(g(x_i) - f(x_i)\right)\Delta x$$

limit of Riemann Sums.

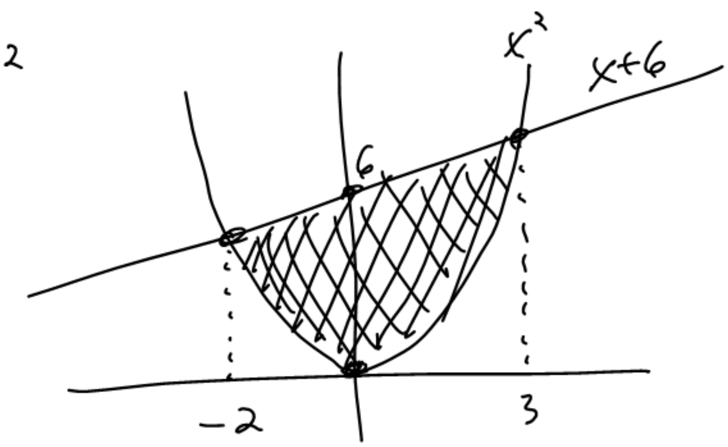
$$= \int_a^b g(x) - f(x)\, dx$$

by definition

example

Consider the line $y = x+6$ and the parabola $y = x^2$

Calculate the area enclosed between the two.



Let's find the intersection pts.

$$x^2 = x+6$$

$$x^2 - x - 6 = 0$$

$$(x-3)(x+2) = 0$$

$$x = -2, 3$$

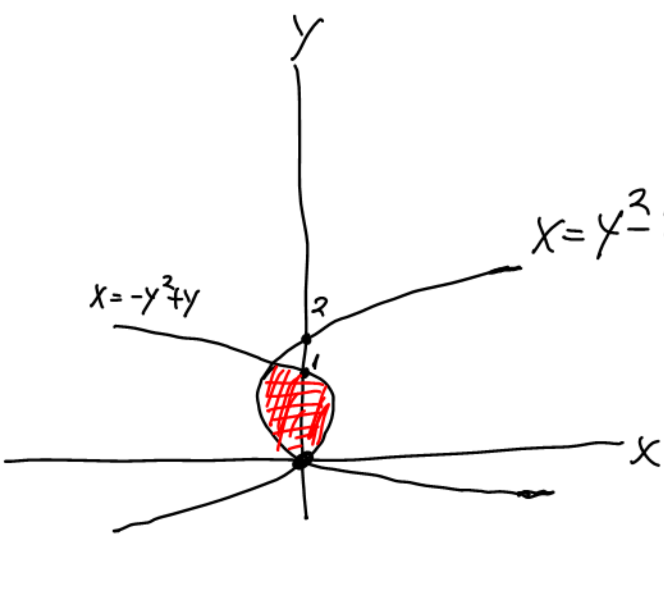
$$\text{Area} = \int_{-2}^{3} (x+6-x^2)\,dx = \frac{1}{2}x^2 + 6x - \frac{1}{3}x^3 \Big]_{-2}^{3}$$

$$= \frac{1}{2}9 + 18 - 9 - \left(2 - 12 + \frac{8}{3}\right)$$

$$= 18 - \frac{9}{2} - \left(-10 + \frac{8}{3}\right) = \boxed{\frac{125}{6} \text{ units}^2}$$

example

Calculate the following area



$$-y^2 + y = y^2 - 2y$$

$$0 = 2y^2 - 3y$$

$$0 = y^2 - \frac{3}{2}y$$

$$0 = y\left(y - \frac{3}{2}\right)$$

$$y = 0, \frac{3}{2}$$

$$\int_a^b (\text{right} - \text{left})\,dy = \int_0^{3/2} (-y^2+y) - (y^2-2y)\,dy$$

$$= \int_0^{3/2} -2y^2+3y\,dy = -\frac{2}{3}y^3 + \frac{3}{2}y^2 \Big]_{y=0}^{y=\frac{3}{2}} = -\frac{2}{3}\left(\frac{3}{2}\right)^3 + \frac{3}{2}\left(\frac{3}{2}\right)^2 - 0$$

$$= -\frac{9}{4} + \frac{3}{2}\left(\frac{9}{4}\right)$$

$$= \frac{1}{2}\left(\frac{9}{4}\right) = \frac{9}{8} \text{ units}^2$$