Section 1.7

Recall

Combining these with the chain rule we get the following.

$$\frac{d}{dx}\text{Arcsin}(x)=\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\text{Arcsin}(g(x))=\frac{g'(x)}{\sqrt{1-(g(x))^2}}$$

$$\frac{d}{dx}\text{Arctan}(x)=\frac{1}{1+x^2} \qquad \frac{d}{dx}\text{Arctan}(g(x))=\frac{g'(x)}{1+(g(x))^2}$$

$$\frac{d}{dx}\text{Arcsec}(x)=\frac{1}{|x|\sqrt{x^2-1}} \qquad \frac{d}{dx}\text{Arcsec}(g(x))=\frac{g'(x)}{|g(x)|\sqrt{(g(x))^2-1}}$$

example

$$\int\frac{1}{9+4x^2}\,dx=\frac{1}{9}\int\frac{1}{1+\frac{4}{9}x^2}\,dx=\frac{1}{9}\int\frac{1}{1+\left(\frac{2}{3}x\right)^2}\,dx$$

let $u=\frac{2}{3}x$

$$\frac{du}{dx}=\frac{2}{3}$$

$$\frac{3}{2}du=dx$$

$$\frac{1}{9}\frac{3}{2}\int\frac{1}{1+u^2}\,du=\frac{1}{6}\text{Arctan}(u)+C$$

$$\boxed{=\frac{1}{6}\text{Arctan}\left(\frac{2}{3}x\right)+C}$$

This works out for any $\displaystyle\int\frac{1}{a^2+b^2x^2}\,dx$

example

given any quadratic polynomial ax^2+bx+c such that

$\sqrt{b^2-4ac}$ is imaginary

$\int \frac{1}{ax^2+bx+c}\,dx$ can be solved by completing the square.

and using $\int \frac{1}{1+u^2}\,du = \text{Arctan}(u)+c.$

$$\int \frac{1}{x^2+x+\frac{5}{4}}\,dx \quad \int \frac{1}{x^2+x+\frac{1}{4}-\frac{1}{4}+\frac{5}{4}}\,dx = \int \frac{1}{\left(x^2+x+\frac{1}{4}\right)+1}\,dx = \int \frac{1}{\left(x+\frac{1}{2}\right)^2+1}\,dx$$

coefficient of x is 1,

$\left(\frac{1}{2}\right)^2 = \frac{1}{4}$

let $u = x+\frac{1}{2}$

$du = dx$

$$= \int \frac{1}{1+u^2}\,du = \text{Arctan}(u) + C = \boxed{\text{Arctan}\left(x+\frac{1}{2}\right)+C}$$

Some other examples.

(1)
$$\int_{\frac{1}{2}}^{1} \frac{\operatorname{Arcsin}(x)}{\sqrt{1-x^2}}\,dx = \int_{\frac{1}{2}}^{1} \underbrace{\operatorname{Arcsin}(x)}_{g(x)} \underbrace{\frac{1}{\sqrt{1-x^2}}}_{g'(x)}\,dx = \int_{x=\frac{1}{2}}^{x=1} u \frac{1}{\cancel{\sqrt{1-x^2}}} \cancel{\sqrt{1-x^2}}\,du$$

let $u = \operatorname{Arcsin}(x)$

$$\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2}\,du = dx$$

$$= \int_{x=\frac{1}{2}}^{x=1} u\,du = \frac{1}{2}u^2 \Big]_{x=\frac{1}{2}}^{x=1} = \frac{1}{2}\left(\operatorname{Arcsin}(1)\right)^2 - \frac{1}{2}\left(\operatorname{Arcsin}\left(\tfrac{1}{2}\right)\right)^2$$

$$\boxed{= \frac{1}{2}\left(\frac{\pi}{6}\right)^2 - \frac{1}{2}\left(\frac{\pi}{3}\right)^2}$$