Section 1.6 Integrals/antiderivatives using logs and exponentials.

Exponentials

Recall that $\frac{d}{dx}e^x = e^x$, Therefore $\int e^x dx = e^x + C$.

Combining with the chain rule we obtain the following

$$\frac{d}{dx}e^{g(x)} = e^{g(x)}g'(x) = g'(x)e^{g(x)}$$

Therefore

$$\int g'(x)e^{g(x)}dx = e^{g(x)} + C.$$

normally this is solved by making the substitution let $\boxed{u = g(x)}$

Then $\frac{du}{dx} = g'(x)$

$$\frac{du}{g'(x)} = dx$$

<u>example</u> Calculate

$$\int x^2 e^{-x^3} dx = \int \cancel{x^2} e^u \frac{du}{-3\cancel{x^2}} = -\frac{1}{3}\int e^u du = -\frac{1}{3}e^u + C$$

let $u = -x^3$

$\frac{du}{dx} = -3x^2$

$\frac{du}{-3x^2} = dx$

$$\boxed{= -\frac{1}{3}e^{-x^3} + C}$$

<u>Remark</u> If instead we were presented with $\int x e^{-x^3} dx$ Then This substitution technique wouldn't work because $g(x) = -x^3$ and $g'(x) = -3x^2$ which is not a constant multiple of x.

<u>example</u>

$$\int_{2/3}^{1} e^{2-3x} dx = \int_{x=\frac{2}{3}}^{x=1} e^u \frac{du}{-3} = -\frac{1}{3}\int_{u=0}^{u=-1} e^u du = -\frac{1}{3} e^u \Big]_{u=0}^{u=-1}$$

$u = 2-3x$

$\frac{du}{dx} = -3$

$\frac{du}{-3} = dx$

$$= -\frac{1}{3}\left(e^{-1} - e^0\right)$$

$$\boxed{= \frac{1}{3}\left(1 - \frac{1}{e}\right)}$$

alternative calculation

$$\int_{2/3}^{1} e^{2-3x}\,dx = \int_{x=\frac{2}{3}}^{x=1} e^{u}\,\frac{du}{-3} = -\frac{1}{3}\int_{x=\frac{2}{3}}^{x=1} e^{u}\,du = -\frac{1}{3}\,e^{u}\Big]_{x=\frac{2}{3}}^{x=1}$$

$u = 2-3x$

$\frac{du}{dx} = -3$

$\frac{du}{-3} = dx$

$$= -\frac{1}{3}\,e^{2-3x}\Big]_{2/3}^{1}$$

$$= -\frac{1}{3}\left(e^{-1} - e^{0}\right)$$

$$\boxed{= \frac{1}{3}\left(1-\frac{1}{e}\right)}$$

example $\boxed{\text{Remember } e^{\ln(a)} = a \quad \text{and} \quad (e^{s})^{t} = e^{st}}$

Calculate $\int 2^{x}\,dx = \int \left(e^{\ln(2)}\right)^{x} dx = \int e^{\ln(2)x}\,dx =$

let $u = \ln(2)\,x$

$\frac{du}{dx} = \ln(2)$

$\frac{du}{\ln(2)} = dx$

$$\int e^{u}\,\frac{du}{\ln(2)} = \frac{1}{\ln(2)}\int e^{u}\,du = \frac{1}{\ln(2)}\,e^{u} + C = \frac{1}{\ln(2)}\,e^{\ln(2)x} + C$$

$$\boxed{= \frac{1}{\ln(2)}\,2^{x} + C}$$

Logarithms

Recall that $\frac{d}{dx}\ln(x) = \frac{1}{x}$

Therefore $\int \frac{1}{x}\,dx = \ln|x| + C$ The absolute value is because $\frac{1}{x}$ is defined for any $x \neq 0$ while $\ln(x)$ is only defined for $x > 0$.

Combining this with the chain rule

$$\frac{d}{dx}\ln(g(x)) = \frac{1}{g(x)}\,g'(x) = \frac{g'(x)}{g(x)}.$$

Therefore $\int \frac{g'(x)}{g(x)}\,dx = \ln|g(x)| + C.$

example

$$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \underbrace{\frac{2x}{x^2+1}}_{\frac{g'}{g}}\,dx = \frac{1}{2}\ln|x^2+1| + C = \boxed{\frac{1}{2}\ln(x^2+1) + C}$$

example

$$\int \tan(\theta)\,d\theta = \int \frac{\sin(\theta)}{\cos(\theta)}\,d\theta = -\int \frac{-\sin(\theta)}{\cos(\theta)}\,d\theta = \boxed{-\ln|\cos(\theta)| + C}$$

$$\frac{g'(x)}{g(x)}$$

$$= \ln|\cos(\theta)^{-1}| + C$$

$$= \ln\left|\frac{1}{\cos(\theta)}\right| + C$$

$$\boxed{= \ln|\sec(\theta)| + C}$$

example $$\int \frac{1}{2x+1}\,dx = \frac{1}{2}\int \frac{2}{2x+1}\,dx = \boxed{\frac{1}{2}\ln|2x+1| + C}$$

example $$\int_1^2 \frac{(\ln(x))^2}{x}\,dx = \int_{x=1}^{x=2} \frac{u^2}{\cancel{x}}\,\cancel{x}\,du = \int_{x=1}^{x=2} u^2\,du = \frac{1}{3}u^3\Big]_{x=1}^{x=2}$$

let $u = \ln(x)$

$$\frac{du}{dx} = \frac{1}{x}$$

$$x\,du = dx$$

$$= \frac{1}{3}(\ln(x))^3\Big]_{x=1}^{x=2}$$

$$= \frac{1}{3}\left(\ln(2)^3 - 0\right)$$

$$\boxed{= \frac{(\ln(2))^3}{3}}$$