Math2310 – Summer2023 – Exam 1 **Name**____________________

(1)[10pts] Calculate $\int_0^1 x^2 e^{1-x^3}\,dx$

let $u = 1 - x^3$

$\frac{du}{dx} = -3x^2$

$\frac{du}{-3x^2} = dx$

$$\int_0^1 x^2 e^{1-x^3}\,dx = \int_{x=0}^{x=1} \cancel{x^2} e^u \frac{du}{-3\cancel{x^2}} = -\frac{1}{3}\int_{x=0}^{x=1} e^u\,du = -\frac{1}{3}e^u\Big]_{x=0}^{x=1} = -\frac{1}{3}e^{1-x^3}\Big]_0^1$$

$$= -\frac{1}{3}\left(e^0 - e^1\right) = \boxed{\frac{e-1}{3}}$$

(2)[10pts] Calculate $\int \frac{\cos(3x) - \sin(3x)}{\sin(3x) + \cos(3x)}\,dx$

$$\int \frac{\cos(3x) - \sin(3x)}{\sin(3x) + \cos(3x)}\,dx = \frac{1}{3}\int \frac{3\cos(3x) - 3\sin(3x)}{\sin(3x) + \cos(3x)}\,dx$$

$\frac{g'(x)}{g(x)}$

$$\boxed{= \frac{1}{3}\ln\left|\sin(3x) + \cos(3x)\right| + C}$$

(3)[10pts] Calculate $\int \frac{(\ln(x))^2}{x}dx$ $= \int \frac{u^2}{\cancel{x}} \cancel{x}\, du = \int u^2 du = \frac{1}{3}u^3 + C$ (2)

let $u = \ln(x)$ (2)

$\frac{du}{dx} = \frac{1}{x}$

$x\,du = dx$

(3)

$$\boxed{= \frac{1}{3}\left(\ln(x)\right)^3 + C}$$

(3)

(4)[10pts] Calculate $\int_0^2 \frac{1}{\sqrt{16-x^2}}dx$ $= \int_0^2 \frac{1}{\sqrt{16\left(1-\frac{x^2}{16}\right)}}dx = \frac{1}{4}\int_0^2 \frac{1}{\sqrt{1-\left(\frac{x}{4}\right)^2}}dx$

(2) (2)

let $u = \frac{x}{4}$ (2)

$\frac{du}{dx} = \frac{1}{4}$

$4\,du = dx$

$$= \frac{1}{\cancel{4}}\int_{x=0}^{x=2} \frac{1}{\sqrt{1-u^2}} \cancel{4}\, du = \operatorname{Arcsin}(u)\Big]_{x=0}^{x=2} = \operatorname{Arcsin}\left(\frac{x}{4}\right)\Big]_0^2$$

(2)

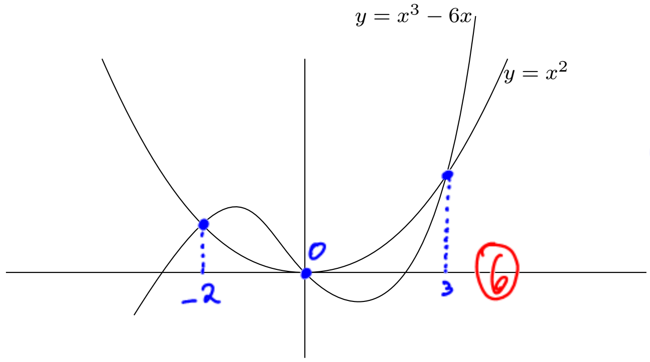
$$= \operatorname{Arcsin}\left(\frac{1}{2}\right) - \operatorname{Arcsin}(0)$$

$$= \frac{\pi}{6} - 0$$

$$= \frac{\pi}{6}$$

(2)

(5)[18pts] The graphs $y = x^2$ and $y = x^3 - 6x$ are as shown. What is the area of the finite regions enclosed between the two curves.



$x^2 = x^3 - 6x$

$0 = x^3 - x^2 - 6x$

$0 = x(x-3)(x+2)$

$x = 0, 3, -2$

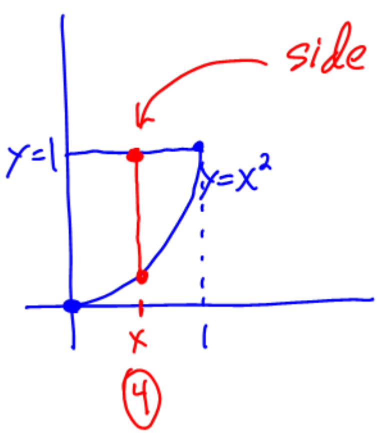
$$\text{Area} = \int_{-2}^{0} x^3 - 6x - x^2\,dx + \int_{0}^{3} x^2 - x^3 + 6x\,dx = \left.\tfrac{1}{4}x^4 - 3x^2 - \tfrac{1}{3}x^3\right]_{-2}^{0} + \left.\tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + 3x^2\right]_{0}^{3}$$

8

4

$$= 0 - \left(4 - 12 + \tfrac{8}{3}\right) + \left(9 - \tfrac{81}{4} + 27\right) - 0 = 44 - \tfrac{8}{3} - \tfrac{81}{4} = \frac{528 - 32 - 243}{12} = \frac{253}{12}$$

(6)[14pts] The base of a solid is the region in the first quadrant of the xy-plane enclosed by the parabola $y = x^2$ and the horizontal line $y = 1$. Slices of the solid perpendicular to the x-axis are squares. Calculate the volume of this solid.



side length $= 1 - x^2$

$A(x) = (1-x^2)^2$ 4

4

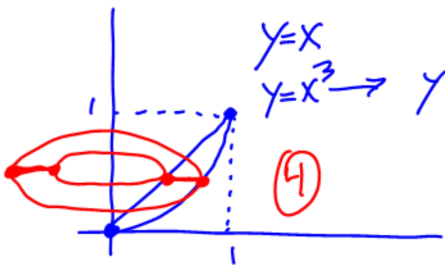
$$\text{Volume} = \int_0^1 A(x)\,dx = \int_0^1 (1-x^2)^2\,dx$$

$$= \int_0^1 x^4 - 2x^2 + 1\,dx = \left.\tfrac{1}{5}x^5 - \tfrac{2}{3}x^3 + x\right]_0^1 = \tfrac{1}{5} - \tfrac{2}{3} + 1$$ 2

$$= \frac{3 - 10 + 15}{15} = \frac{8}{15}$$

(7)[28pts] Consider the region of the first quadrant of the xy-plane enclosed between the curves $y = x^3$ and $y = x$. Consider the solid obtained by rotating this region around the y-axis.

(a) Calculate an integral which uses slices perpendicular to the y-axis to measure the volume of the solid.



$y=x$

$y=x^3 \rightarrow y^{\frac{1}{3}}=x$

(4)

$$\text{Volume}=\pi\int_0^1 y^{\frac{2}{3}}-y^2\,dy=\pi\left(\frac{3}{5}y^{\frac{5}{3}}-\frac{1}{3}y^3\right)\Big]_0^1$$

(4)

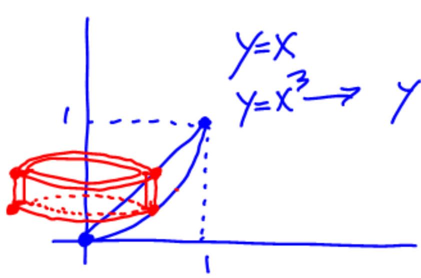
$$A(x)=\pi R^2-\pi r^2$$

$$=\pi\left(\left(y^{\frac{1}{3}}\right)^2-y^2\right) \quad (4)$$

$$=\pi\left(\frac{3}{5}-\frac{1}{3}\right)$$

(2) $$=\boxed{\frac{4\pi}{15}}$$

(b) Calculate an integral which uses cylindrical shells centered on the y-axis to measure the volume of the solid.



$y=x$

$y=x^3 \rightarrow y^{\frac{1}{3}}=x$

$$\text{Volume}=2\pi\int_0^1 x\,(\text{shell height})\,dx \quad (6)$$

$$=2\pi\int_0^1 x(x-x^3)\,dx \quad (4)$$

$$=2\pi\int_0^1 x^2-x^4\,dx$$

$$=2\pi\left(\frac{1}{3}x^3-\frac{1}{5}x^5\right)\Big]_0^1=2\pi\left(\frac{1}{3}-\frac{1}{5}\right)=\boxed{\frac{4\pi}{15}} \quad (2)$$

Same answer (4)