# 3.5 Derivatives of trig functions

Consider the following limit

$$\lim_{x\to 0} \frac{\sin(x)}{x}$$

This limit has the $\frac{0}{0}$ form.

| $x$ | $\frac{\sin(x)}{x}$ |
|---|---|
| 0.1 | .998334---- |
| 0.01 | .999983---- |
| -0.01 | .999983 --- |
| -0.1 | .998334--- |

It seems that

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1.$$

There is an actual geometric proof of this fact, but we'll leave it out.

Angle sum formula

$\sin(A+B) = \sin(A)\cos(B) + \sin(B)\cos(A)$

Now, by definition

$$\frac{d}{dx}\sin(x) = \lim_{h\to 0} \frac{\sin(x+h)-\sin(x)}{h} = \lim_{h\to 0} \frac{\sin(x)\cos(h)+\sin(h)\cos(x) - \sin(x)}{h}$$

$$= \lim_{h\to 0}\left[\frac{\sin(x)\cos(h)-\sin(x)}{h} + \frac{\sin(h)\cos(x)}{h}\right]$$

$$= \lim_{h\to 0}\left[\sin(x)\frac{\cos(h)-1}{h} + \cos(x)\frac{\sin(h)}{h}\right]$$

Similar to $\lim_{x\to 0}\frac{\sin(x)}{h} = 1$ It is also true that $\lim_{x\to 0}\frac{\cos(x)-1}{x} = 0$

$$= \sin(x)(0) + \cos(x)(1)$$

$$\boxed{= \cos(x)}$$

Thus $\frac{d}{dx}\sin(x) = \cos(x)$

Similarly $\frac{d}{dx}\cos(x) = -\sin(x)$

here's the proof

$$\frac{d}{dx}\cos(x) = \lim_{h\to 0}\frac{\cos(x+h)-\cos(x)}{h}$$

$$= \lim_{h\to 0}\frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h}$$

$$= \lim_{h\to 0}\left[\frac{\cos(x)\cos(h)-\cos(x)}{h} - \frac{\sin(x)\sin(h)}{h}\right]$$

$$= \lim_{h\to 0}\left[\cos(x)\frac{\cos(h)-1}{h} - \sin(x)\frac{\sin(h)}{h}\right]$$

$$= \cos(x)(0) - \sin(x)(1) = \boxed{-\sin(x)}$$

$$\frac{d}{dx}\tan(x) = \sec^2(x)$$

here's the proof

$$\frac{d}{dx}\tan(x) = \frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right) = \frac{\cos(x)\cos(x) - (-\sin(x))\sin(x)}{\cos^2(x)}$$

$$= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)}$$

$$= \frac{1}{\cos^2(x)}$$

$$= \left(\frac{1}{\cos(x)}\right)^2$$

$$= \boxed{\sec^2(x)}$$

Summary

$$\frac{d}{dx}\sin(x) = \cos(x)$$

$$\frac{d}{dx}\cos(x) = -\sin(x)$$

$$\frac{d}{dx}\tan(x) = \sec^2(x)$$

$$\frac{d}{dx}\cot(x) = -\csc^2(x)$$

$$\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$$

$$\frac{d}{dx}\csc(x) = -\csc(x)\cot(x).$$