2.3 Calculating limits with "limit laws"

Some properties of limits

① $\lim_{x\to a}\left(f(x)+g(x)\right) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

② $\lim_{x\to a}\left(k f(x)\right) = k \lim_{x\to a} f(x)$

③ $\lim_{x\to a}\left(f(x)\, g(x)\right) = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right)$

④ $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ unless $\lim_{x\to a} g(x) = 0$.

⑤ $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ and $\lim_{x\to a}\left(f(x)\right)^n = \left(\lim_{x\to a} f(x)\right)^n$

⑥ If $f(x)$ is a polynomial, $\sin(x)$, $\cos(x)$, $\sqrt{x}$, e^x

then $\lim_{x\to a} f(x) = f(a)$

examples

$$\lim_{x\to 2} \frac{5x^2-3}{\sqrt{x^3+1}} = \frac{\lim_{x\to 2} 5x^2-3}{\lim_{x\to 2} \sqrt{x^3+1}} = \frac{\lim_{x\to 2} 5x^2-3}{\sqrt{\lim_{x\to 2} x^3+1}} = \frac{20-3}{\sqrt{8+1}} = \frac{17}{\sqrt{9}} = \frac{17}{3}$$

example

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{(x+2)\cancel{(x-2)}}{\cancel{(x-2)}} = \lim_{x\to 2}(x+2) = 4$$

$\frac{0}{0}$ form because top limit and bottom limit are both 0.

"Factor + Cancel" technique

$$\lim_{x\to -1} \frac{\frac{1}{x}+1}{x+1} = \lim_{x\to -1} \frac{\frac{1}{x}+\frac{x}{x}}{\frac{x+1}{1}} = \lim_{x\to -1} \frac{\frac{x+1}{x}}{\frac{x+1}{1}} = \lim_{x\to -1} \frac{\cancel{x+1}}{x}\,\frac{1}{\cancel{x+1}}$$

$\frac{0}{0}$ form "simplifying nested fractions and canceling"

$$= \lim_{x\to -1} \frac{1}{x} = \frac{1}{-1} = -1$$

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b}\frac{d}{c}$$

$\boxed{\text{Remember } (a-b)(a+b) = a^2-b^2}$

example

$$\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9} = \lim_{x\to 9} \frac{\sqrt{x}-3}{x-9}\,\frac{\sqrt{x}+3}{\sqrt{x}+3} = \lim_{x\to 9} \frac{(\sqrt{x})^2-9}{(x-9)(\sqrt{x}+3)}$$

$\frac{0}{0}$ form

"Conjugation technique"

fraction equals 1 and multiplying by 1 doesn't change the value.

$$= \lim_{x\to 9} \frac{\cancel{x-9}}{\cancel{(x-9)}(\sqrt{x}+3)} = \lim_{x\to 9} \frac{1}{\sqrt{x}+3} = \frac{1}{\sqrt{9}+3} = \frac{1}{6}$$

$\frac{k}{0}$ form limits

When $\lim_{x \to a} f(x) = k \neq 0$ and $\lim_{x \to a} g(x) = 0$,

Then $\lim_{x \to a} \frac{f(x)}{g(x)}$ has only 3 possibilities,

$+\infty$, $-\infty$, or does not exist.

If as $x \to a$ $\frac{f(x)}{g(x)}$ is positive, Then $\lim_{x \to a} \frac{f(x)}{g(x)} = +\infty$

If as $x \to a$ $\frac{f(x)}{g(x)}$ is negative, Then $\lim_{x \to a} \frac{f(x)}{g(x)} = -\infty$

If as $x \to a$ $\frac{f(x)}{g(x)}$ is both positive and negative, Then $\lim_{x \to a} \frac{f(x)}{g(x)}$ does not exist.

example

$$\lim_{x \to 2^+} \frac{3x+5}{x-2}$$

top limit $= 11 \neq 0$

bottom limit $= 0$

$\frac{\text{non zero}}{0}$ form.

as $x \to 2^+$

$x > 2$, Therefore

$3x+5$ is positive,

$x-2$ is positive

So $\frac{3x+5}{x-2}$ is positive

Therefore

$$\lim_{x \to 2^+} \frac{3x+5}{x-2} = +\infty$$

Similarly $\lim_{x \to 2^-} \frac{3x+5}{x-2} = -\infty$ so $\lim_{x \to 2} \frac{3x+5}{x-2}$ does not exist.