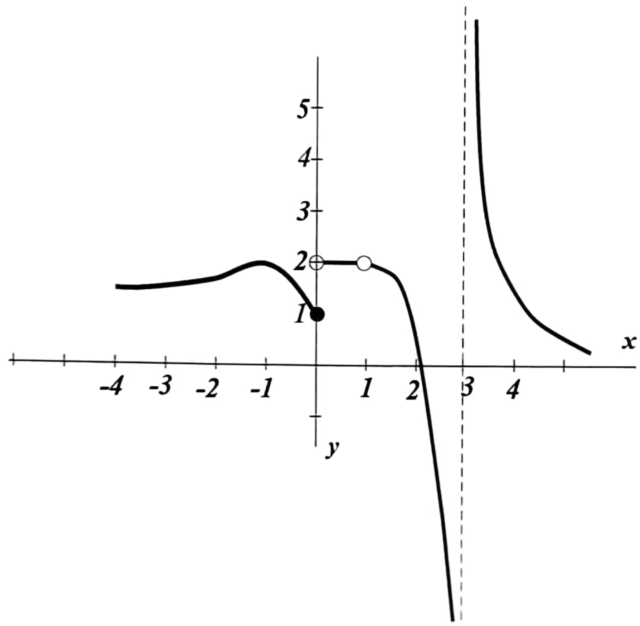
Math2300 – Summer2023 – Exam 1 Name: Solutions

(1)[18pts] Find the following limits and answer the following questions for the function $f(x)$ whose graph is shown.



$\lim_{x \to 0^+} f(x) = 2$

$\lim_{x \to 0^-} f(x) = 1$

$\lim_{x \to -1} f(x) = 2$

$\lim_{x \to 3^-} f(x) = -\infty$

$\lim_{x \to 1} f(x) = 2$

Where is $f(x)$ not continuous on $-4 \le x \le 5$?

$X = 0, 1, 3$

(2)[24pts] Calculate the following three limits *precisely*, no approximations. Your answers must be properly written.

$$\lim_{x\to 4} \frac{\sqrt{x}-2}{x(x-4)} = \lim_{x\to 4} \frac{\sqrt{x}-2}{x(x-4)} \frac{\sqrt{x}+2}{\sqrt{x}+2} = \lim_{x\to 4} \frac{\cancel{x-4}}{x(\cancel{x-4})(\sqrt{x}+2)}$$

$$= \lim_{x\to 4} \frac{1}{x(\sqrt{x}+2)} = \frac{1}{4(2+2)} = \frac{1}{16}$$

$$\lim_{x\to 1} \frac{\frac{1}{x+1}-\frac{1}{2}}{1-x} = \lim_{x\to 1} \frac{\frac{2}{2(x+1)} - \frac{x+1}{2(x+1)}}{\frac{1-x}{1}} = \lim_{x\to 1} \frac{\cancel{1-x}}{2(x+1)} \frac{1}{\cancel{1-x}} =$$

$$= \lim_{x\to 1} \frac{1}{2(x+1)} = \frac{1}{2(1+1)} = \frac{1}{4}$$

(3)[14pts] Find $\lim_{x\to 2^+} f(x)$ and $\lim_{x\to 2^-} f(x)$. For which values of a is the function $f(x)$ continuous at $x = 2$.

$$f(x) = \begin{cases} 3x - a & \text{for } x \le 2 \\ ax^2 & \text{for } x > 2 \end{cases}$$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}(3x-a) = 6-a$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} ax^2 = 4a$$

$6-a=4a$

$6=5a$

$\boxed{\frac{6}{5}=a}$ makes $f(x)$ continuous at $x=2$

(4)[16pts] Let $f(x) = x^2 + x - 1$. Use the definition of the derivative to calculate $f'(x)$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2+(x+h)-1-(x^2+x-1)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^2}+2xh+h^2+\cancel{x}+h\cancel{-1}-\cancel{x^2}-\cancel{x}+\cancel{1}}{h} = \lim_{h\to 0} \frac{2xh+h^2+h}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(2x+h+1)}{\cancel{h}} = \lim_{h\to 0} (2x+h+1) = \boxed{2x+1}$$

(5)[14pts] A tank of water is leaking. The volume of water in the tank is $V(t) = 100 - t^2$ gallons at $t \geq 0$ minutes. What are the units of measurement of $V'(t)$? Find the average rate of change of water volume over the interval $8 \leq t \leq 10$ and also over the interval $9 \leq t \leq 10$. Should your average rates of change be positive or negative?

Avg rate of change $8 \leq t \leq 10$ $= \dfrac{V(10) - V(8)}{10-8} = \dfrac{0-36}{2} = -18$ gal/min

Avg rate of change $9 \leq t \leq 10$ $= \dfrac{V(10) - V(9)}{10-9} = \dfrac{-19}{1} = -19$ gal/min

They should be negative because volume $V(t)$ is decreasing.

(6)[14pts] The graph of a function $y = f(x)$ is shown. On the same set of axes sketch a graph of what $y = f'(x)$ should look like.