Section 3.3 Division of Polynomials

Remember back to learning about "long division" of numbers base-10

Division of polynomials is very much the same because base-10 numbers are really just polynomials.

e.g. $21,305 = 2\times10^4 + 1\times10^3 + 3\times10^2 + 0\times10^1 + 5\times10^0$

just like $2x^4 + x^3 + 3x^2 + 5$

<u>example</u> divide $x^3 - x^2 - 14x + 24$ by $x+3$.

divisor has degree 1

$$
\begin{array}{r l}
 & x^2 - 4x - 2 \quad \leftarrow \text{quotient} \\
x+3 \,\big) & x^3 - x^2 - 14x + 24 \\
 & -(x^3 + 3x^2) \\
\hline
 & -4x^2 - 14x + 24 \quad \leftarrow \text{first remainder, degree 2} \\
 & -(-4x^2 - 12x) \\
\hline
 & -2x + 24 \quad \leftarrow \text{second remainder degree 1} \\
 & -(-2x - 6) \\
\hline
 & 30 \quad \leftarrow \text{Third remainder. degree 0 < degree 1 } (x+3)
\end{array}
$$

So this is the final remainder.

final remainder

In the end we obtain

$$\frac{x^3 - x^2 - 14x + 24}{x+3} = (x^2 - 4x - 2) + \frac{30}{x+3}$$

quotient — remainder/divisor.

example divide $x^3-x^2-14x+24$ by x^2+2x+1

$$
\begin{array}{r}
x-3 \qquad \\
x^2+2x+1 \;\big)\; x^3-x^2-14x+24 \\
-(x^3+2x^2+x) \qquad\; \\
\hline
-3x^2-15x+24 \\
-(-3x^2-6x-3) \\
\hline
-9x+27
\end{array}
$$

$x-3 \leftarrow$ quotient

$-3x^2-15x+24 \leftarrow$ first remainder, degree 2

$-9x+27 \leftarrow$ second remainder, degree $1 <$ degree 2 for x^2+2x+1

final remainder

So $$\frac{x^3-x^2-14x+24}{x^2+2x+1} = (x-3) + \frac{-9x+27}{x^2+2x+1}.$$

Final example final remainder is zero

$$
\begin{array}{r}
x^2+4x-2 \qquad \\
x-3 \;\big)\; x^3-x^2-14x+6 \\
-(x^3-3x^2) \qquad\qquad \\
\hline
4x^2-14x+6 \\
-(4x^2-12x) \qquad \\
\hline
-2x+6 \\
-(-2x+6) \\
\hline
0
\end{array}
$$

$0 \leftarrow$ final remainder

So $$\frac{x^3-x^2-14x+6}{x-3} = x^2+2x-2$$

which also means $$x^3-x^2-14x+6 = (x^2+2x-2)(x-3)$$

does this factor further, no.

Factor Theorem

If $f(x)$ is a polynomial and $f(r) = 0$,
Then $x - r$ divides $f(x)$ evenly, i.e., remainder 0.

example $f(x) = x^3 + 2x^2 - 9x - 18$ check $r = -2$

Well $f(-2) = (-2)^3 + 2(-2)^2 - 9(-2) - 18 = -8 + 8 + 18 - 18 = 0$

So $x - (-2) = x + 2$ should divide into $f(x)$ evenly

$$
\begin{array}{r}
x^2 - 9 \\
x+2 \,\overline{\big)\, x^3 + 2x^2 - 9x - 18} \\
\underline{-(x^3 + 2x^2)} \\
-9x - 18 \\
\underline{-9x - 18} \\
0
\end{array}
$$

So
$$\frac{x^3 + 2x^2 - 9x - 18}{x + 2} = x^2 - 9$$

So
$$x^3 + 2x^2 - 9x - 18 = (x^2 - 9)(x + 2)$$
$$x^3 + 2x^2 - 9x - 18 = (x - 3)(x + 3)(x + 2)$$

Rational Root Theorem

If $f(x) = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2x^2 + a_1x + a_0$

coefficient 1   all other coefficients are integers.

Then any other integer root of $f(x)$ is a divisor of a_0.

example

$$x^3+2x^2-9x-18 = (x-3)(x+3)(x+2)$$

This polynomial has roots $3, -3, -2$ which are all divisors of 18.

example Factor fully $f(x) = x^3+4x^2+x-6$

Notice, grouping doesn't work, $x^2(x+4) + (x-6)$

Let's guess at some roots: $\pm 1, \pm 2, \pm 3, \pm 6$.

$r=2$

$f(2) = 8+16+2-6 \neq 0$

$r=1$

$f(1) = 1+4+1-6 = 0$

$$\begin{array}{r|l} & x^2+5x+6 \\ \hline x-1 & x^3+4x^2+x-6 \\ & -(x^3-x^2) \\ \hline & 5x^2+x-6 \\ & -(5x^2-5x) \\ \hline & 6x-6 \\ & -(6x-6) \\ \hline & 0 \end{array}$$

So

$$x^3+4x^2+x-6 = (x-1)(x^2+5x+6)$$

$$\boxed{x^3+4x^2+x-6 = (x-1)(x+3)(x+2)}$$