Section 3.1 Quadratic Functions

A quadratic function is of the form

$$f(x) = ax^2 + bx + c$$

Note: You cannot divide by a on each side without changing the function. This is unlike a quadratic equation $\quad 0 = ax^2 + bx + c$

We can divide both sides of this equation by a and get

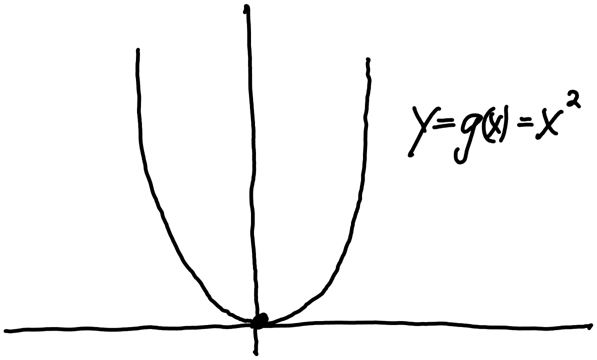
$$\frac{1}{a} 0 = \frac{1}{a}(ax^2 + bx + c)$$

$$0 = x^2 + \frac{b}{a}x + \frac{c}{a}$$

The standard form for a quadratic function is

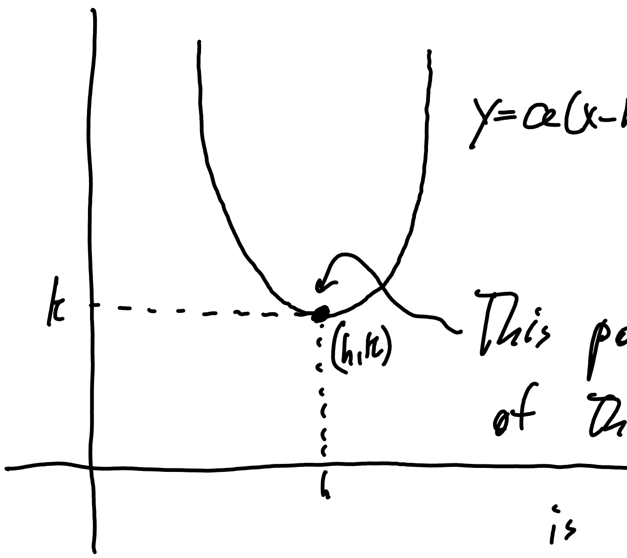
$$f(x) = a(x-h)^2 + k$$

The standard form allows us to immediately graph $y = f(x)$ and see where it's x- and y- intercepts are.



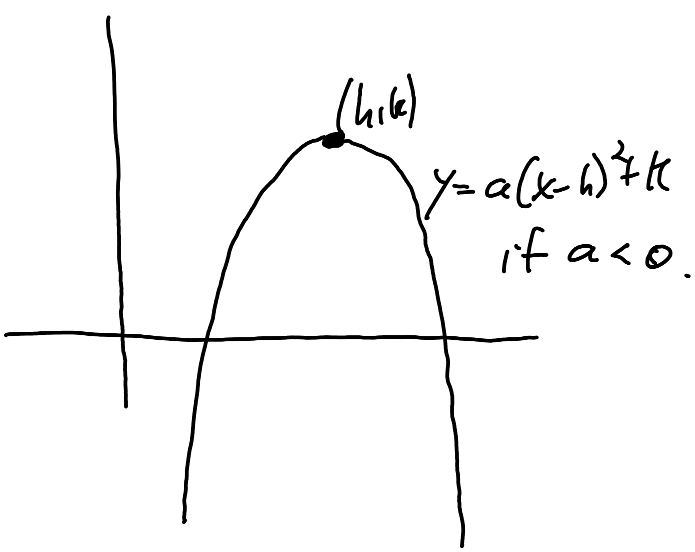
$$f(x)=a(x-h)^2+k$$

$$f(x)=a\,g(x-h)+k$$



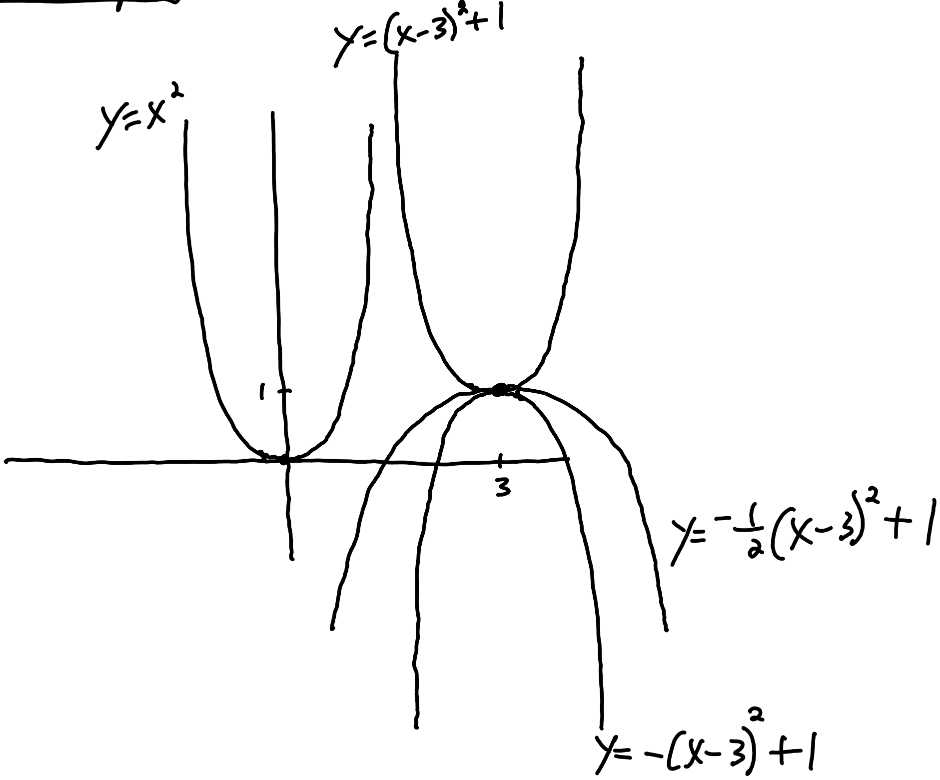
$y=a(x-h)^2+k$ where $h,k,a>0$.

This point is called the "vertex" of the parabola. This point is special because it says that the absolute minimum value of $f(x)=a(x-h)^2+k$ is $y=f(x)=k$ occurring at $x=h$. When $a>0$.



If $a<0$ the the vertex is the absolute maximum for $f(x)$

Some examples



Question how do we go from polynomial form

$$f(x)=ax^2+bx+c$$

to standard form

$$f(x)=a(x-h)^2+k$$

The answer is to complete the square, however there is a twist to it because we can't divide out $a\neq 1$ and just be done with it.

Here is how the process works now

$$f(x) = ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x\right) + c = a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) - \frac{b^2}{4a} + c$$

add $\frac{b^2}{4a^2}$ subtract $\frac{b^2}{4a}$

$$\boxed{\begin{aligned} &= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c \\ &= a\left(x - \left(\frac{-b}{2a}\right)\right)^2 - \frac{b^2}{4a} + c \end{aligned}}$$

standard form.

<u>example</u>

$f(x) = 6x^2 - 60x - 100$ convert this to standard form and determine where the vertex of the parabola is.

$$f(x) = 6x^2 - 60x - 100 = 6(x^2 - 10x) - 100 = 6(x^2 - 10x + 25) - 150 - 100$$

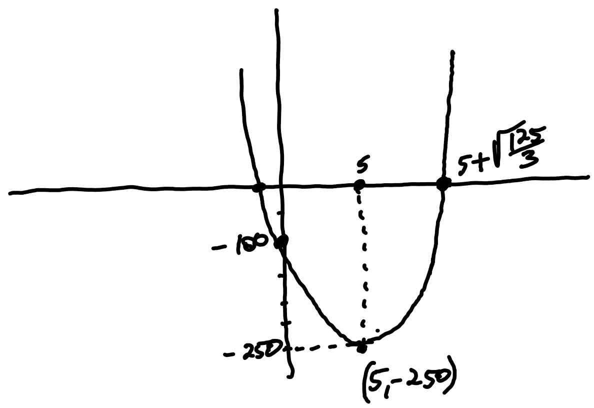
$+25$ $-6 \cdot 25$

$$\boxed{= 6(x-5)^2 - 250}$$

vertex of the parabola is at

$(5, -250)$

Let's graph the function $y = 6(x-5)^2 - 250$



<u>y-intercept</u>

Let $x=0$

$y=6(-5)^2-250=150-250$
$=-100$

<u>x-intercepts</u>

let $y=0$

$0=6(x-5)^2-250$

$\frac{250}{6}=(x-5)^2$

$\frac{125}{3}=(x-5)^2$

$\pm\sqrt{\frac{125}{3}}=x-5$

$5\pm\sqrt{\frac{125}{3}}=x$

<u>Example</u> Let $f(x)=-3x^2-10x+1$. Find the coordinates of the vertex of this parabola.

subtract $-3\frac{25}{9}=-\frac{25}{3}$

$f(x)=-3x^2-10x+1=-3\left(x^2+\frac{10}{3}x\right)+1$

add $\left(\frac{5}{3}\right)^2=\frac{25}{9}$

$=-3\left(x^2+\frac{10}{3}x+\frac{25}{9}\right)-\left(-\frac{25}{3}\right)+1$

$\frac{25}{3}+1$

$=-3\left(x+\frac{5}{3}\right)^2+\frac{28}{3}$

$=-3\left(x-\left(\frac{-5}{3}\right)\right)^2+\frac{28}{3}$ vertex $=\left(\frac{-5}{3},\frac{28}{3}\right)$.

Another way to find the vertex of a parabola.

Let $f(x) = ax^2 + bx + c$

Rather than transforming $f(x)$ into standard form $f(x) = a(x-h)^2 + k$

One can use the fact that $h = \frac{-b}{2a}$ is always true.

Therefore $k = f\left(\frac{-b}{2a}\right)$.

example $f(x) = 6x^2 - 60x - 100 = 6(x-5)^2 - 250$

calculated in the previous example

vertex $= (5, -250)$

Let's confirm this calculation using $a = 6$, $b = -60$, $c = -100$ and $h = \frac{-b}{2a}$.

In this example $h = \frac{-b}{2a} = \frac{-(-60)}{2 \cdot 6} = \frac{60}{12} = 5$ ✓

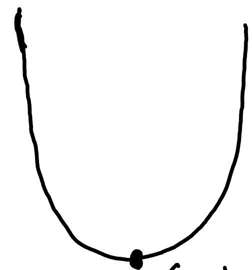
$$k = f\left(\frac{-b}{2a}\right) = f(5) = 6 \cdot 5^2 - 60 \cdot 5 - 100$$

$$= 150 - 300 - 100$$

$$= -250$$ ✓

<u>example</u> $f(x) = 2x^2 + x - 1$. Find the absolute minimum value of $f(x)$.

In other words find the y-coordinate of the vertex of the parabola.



(h,k)

$y = a(x-h)^2 + k$

The $k = f\left(\frac{-b}{2a}\right)$ is the absolute minimum value of $f(x)$.

$f(x) = 2x^2 + x - 1 \qquad a = 2,\ b = 1,\ c = -1$

$$h = \frac{-b}{2a} = \frac{-1}{4}$$

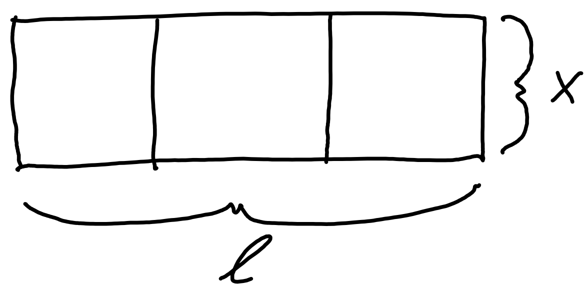
$$k = f\left(\frac{-b}{2a}\right) = f\left(\frac{-1}{4}\right) = 2\left(\frac{-1}{4}\right)^2 + \left(\frac{-1}{4}\right) - 1 = \frac{1}{8} - \frac{1}{4} - 1$$

$$= \frac{-9}{8}$$

absolute minimum value of $f(x)$ occuring at $x = \frac{-1}{4}$.

Modeling with Quadratic functions

example I have 750 ft. of fencing material to build a rectangular pen with 3 compartments as shown



(a) Find a function $A(x)$ for the area enclosed by the pen in ft^2

(b) what is the max area that can be enclosed by my 750 ft. of fencing material?

$A = \text{length} \cdot \text{width} = lx$ but we need l in terms of x.

$750 = 4x + 2l$ solve for l

feet of material

$750 - 4x = 2l$

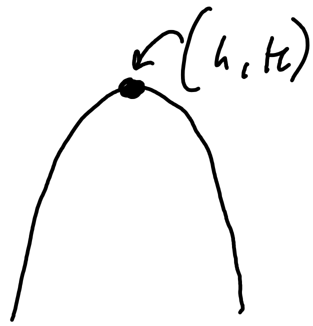
$\underline{375 - 2x = l}$

So now $\boxed{A(x) = (375-2x)x = 375x - 2x^2 \text{ ft}^2}$

(a)

(b) What is the max value of $A(x)$?

$A(x) = -2x^2 + 375x$



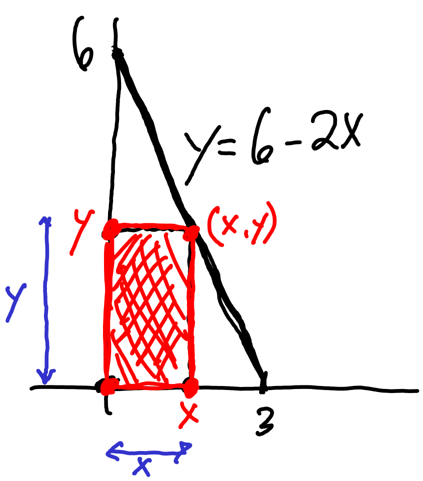
$h = \frac{-b}{2a} = \frac{-375}{2(-2)} = \frac{375}{4} = 93.75$

max value of $A(x)$ occurs at $\boxed{x = 93.75}$

max value of $A(x)$ is $A(93.75) = -2(93.75)^2 + 375(93.75)$

$\boxed{= 17,578.125 \text{ ft}^2}$

<u>example</u> Consider a rectangle inscribed within the triangle as shown.



(a) Find a quadratic function $A(x)$ for the area of the rectangle.

(b) What is the max possible area for such a rectangle?

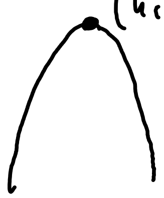
(a) Area = length · width = xy again we must eliminate the y

$y = 6 - 2x$ because (x, y) is on the line $y = 6 - 2x$.

$$A(x) = x(6 - 2x)$$

$$\boxed{A(x) = -2x^2 + 6x \text{ units}^2}$$

(b)



k = max value of $A(x)$

$$h = \frac{-b}{2a} = \frac{-6}{2(-2)} = \frac{6}{4} = \frac{3}{2}$$

$$k = A\left(\frac{3}{2}\right) = -2\left(\frac{3}{2}\right)^2 + 6\left(\frac{3}{2}\right)$$

$$= -2\frac{9}{4} + 9$$

$$= -\frac{9}{2} + 9 = \frac{9}{2} \text{ units}^2$$

max area.