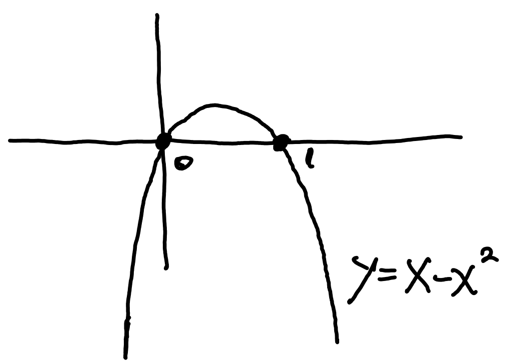
Section 2.8 One-to-one and inverse functions

Definition A function $f(x)$ is one-to-one when

$$f(a) = f(b) \text{ if and only if } a = b$$

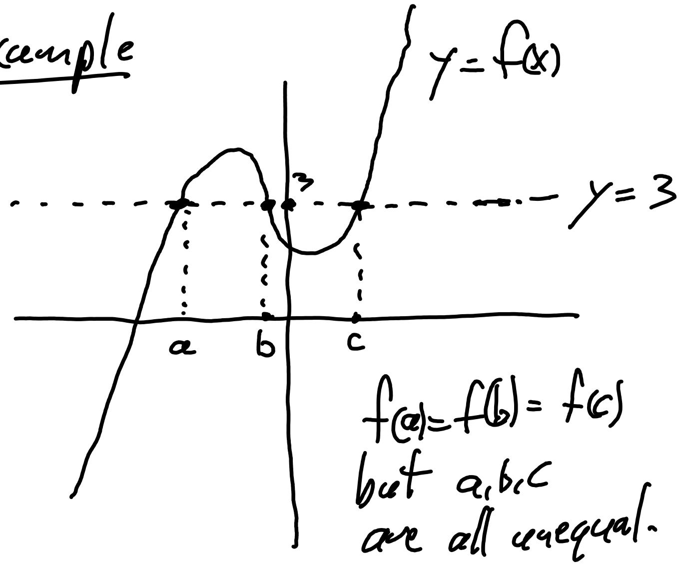
example $f(x) = x - x^2$ is not one-to-one

because $f(0) = 0 = f(1)$ but $1 \neq 0$.



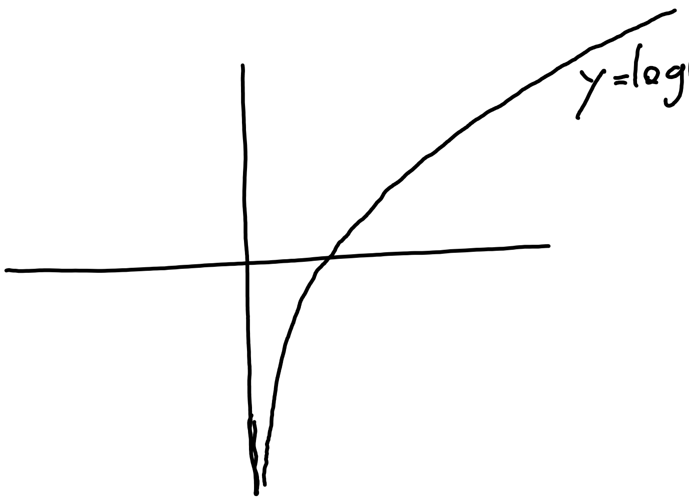
A function $f(x)$ is one-to-one if and only if $y = f(x)$ passes the "horizontal line test"; That is, every horizontal line $y = k$ intersects $y = f(x)$ at most one time.

example

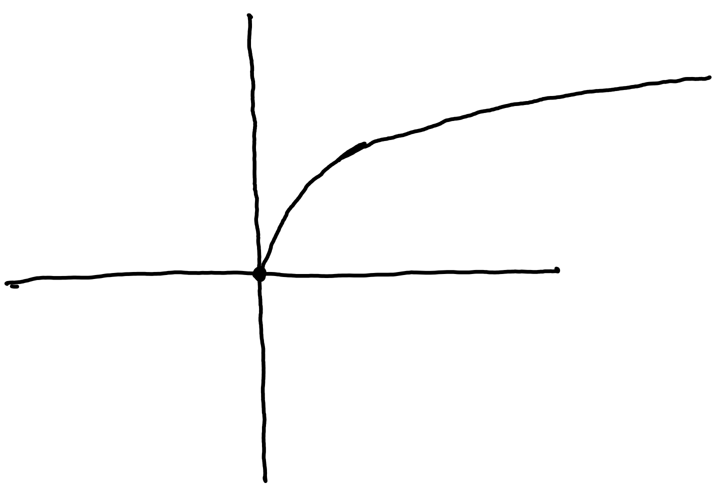


$f(a) = f(b) = f(c)$
but a, b, c
are all unequal.

so $f(x)$ is not one-to-one.



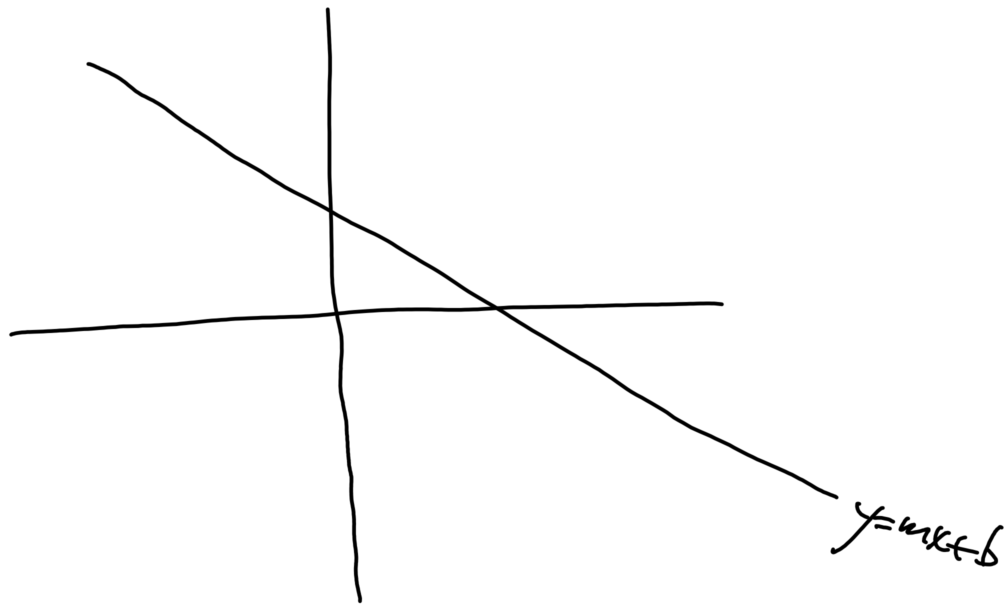
$y = \log(x)$ is one to one.



$y = \sqrt{x}$ is one to one.

Any linear function with $m \neq 0$ is one-to-one

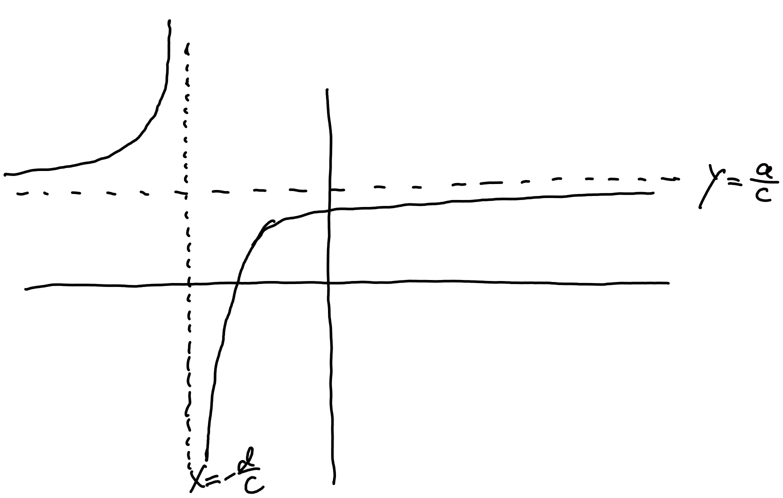
$f(x) = mx + b$

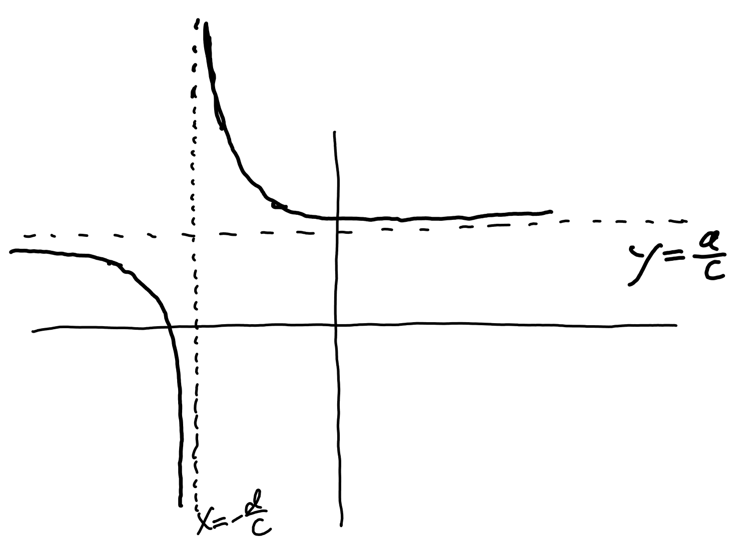


<u>example</u>

A <u>Möbius function</u> has the form $f(x) = \dfrac{ax+b}{cx+d}$

It's graph is as shown and clearly it is one-to-one





Given a one-to-one function $f(x)$ There exists an <u>inverse function</u> $f^{-1}(x)$

Note
$f^{-1}(x)$ doesn't mean $\frac{1}{f(x)}$

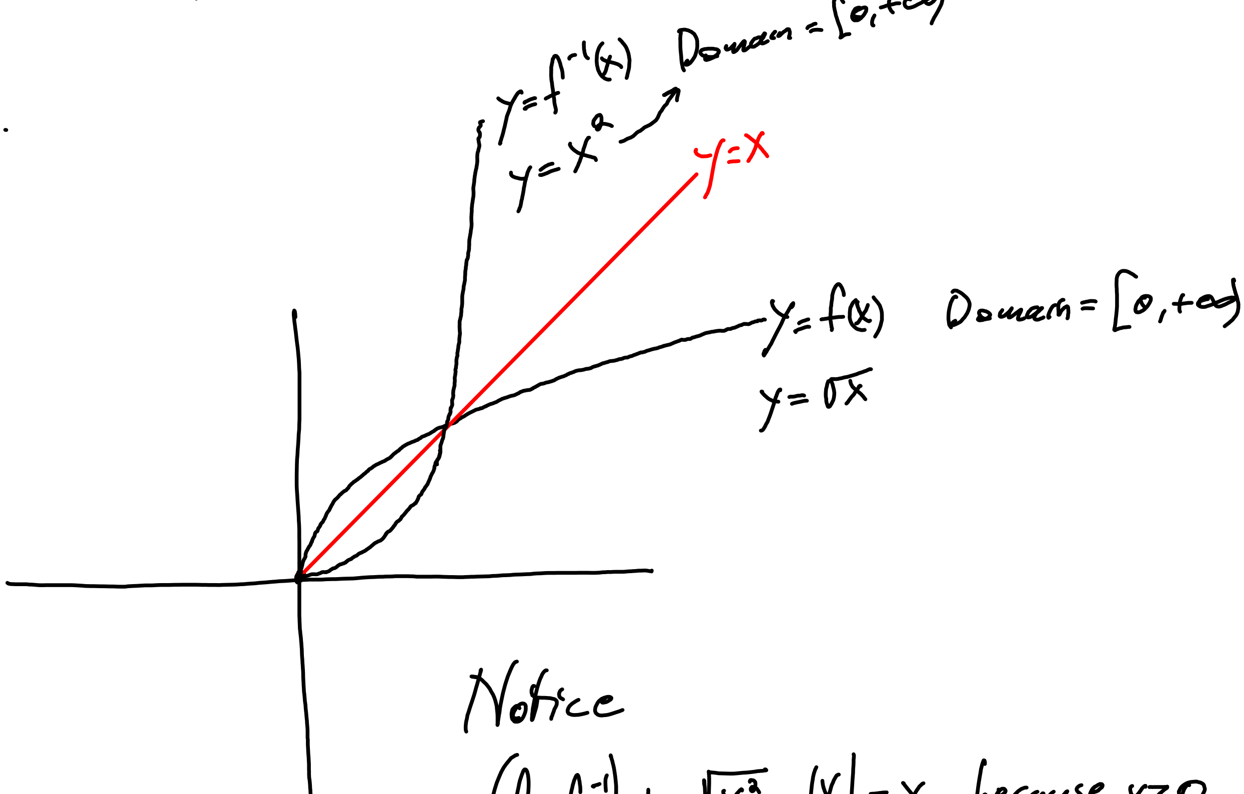
which satisfies

$$(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$$

In other words $b = f(a)$ if and only if $a = f^{-1}(b)$

So the roles of x and y are switched between f and f^{-1} so the graph $y = f^{-1}(x)$ is obtained from $y = f(x)$ by reflection around the line $y = x$.

<u>example</u>



Notice

$(f \circ f^{-1})(x) = \sqrt{x^2} = |x| = x$ because $x \geq 0$.

$(f^{-1} \circ f)(x) = (\sqrt{x})^2 = x$

Finding inverse functions for lines and Möbius functions

example Given linear function $f(x) = 3x - 4$ find $f^{-1}(x)$.

$y = 3x - 4$ switch x and y

$x = 3y - 4$ solve for y.

$x + 4 = 3y$

$\frac{x+4}{3} = y$

$\boxed{\frac{1}{3}x + \frac{4}{3} = y}$

$\boxed{f^{-1}(x) = \frac{1}{3}x + \frac{4}{3}}$

Check

$(f \circ f^{-1})(x) = f(f^{-1}(x)) = f\left(\frac{1}{3}x + \frac{4}{3}\right) = 3\left(\frac{1}{3}x + \frac{4}{3}\right) - 4 = x + 4 - 4 = x$ ✓

$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(3x - 4) = \frac{1}{3}(3x - 4) + \frac{4}{3} = x - \frac{4}{3} + \frac{4}{3} = x$ ✓

example let $f(x) = \frac{3x-4}{x+1}$ find $f^{-1}(x)$.

$y = \frac{3x-4}{x+1}$ swap x and y

$x = \frac{3y-4}{y+1}$ solve for y

LCD $y+1$

$$x(y+1) = \frac{3y-4}{\cancel{y+1}}\cancel{(y+1)}$$

$$x(y+1) = 3y-4$$

$$xy + x = 3y-4$$
$$-3y \quad -3y$$

put y-terms on one side and non-y terms on the other.

$$xy - 3y + x = -4$$
$$-x \quad -x$$

$$xy - 3y = -4 - x$$

$$\frac{(x-3)y}{x-3} = \frac{-x-4}{x-3}$$

$$y = \frac{-x-4}{x-3}$$

$$\boxed{f^{-1}(x) = \frac{-x-4}{x-3}}$$

also a Möbius Function.

<u>example</u> Consider $x,\ 1-x,\ \frac{1}{x},\ \frac{1}{1-x},\ \frac{x}{x-1},\ \frac{x-1}{x}$

If we pick $f(x)$ and $g(x)$ from this list, then $(f\circ g)(x)$ will also be in this list.

In particular each of the six functions has its inverse among the 6 as well.

* $f(x)=x$, $g(x)=x$ now $(f\circ g)(x)=f(g(x))=f(x)=x$

So $f(x)=x$ is its own inverse function.

* $f(x)=1-x$, $g(x)=1-x$, now $(f\circ g)(x)=f(g(x))=f(1-x)=1-(1-x)$
$=x$

So $f(x)=1-x$ is its own inverse function.

* $f(x)=\frac{1}{x}$, $g(x)=\frac{1}{x}$, now $(f\circ g)(x)=f(g(x))=f\left(\frac{1}{x}\right)=\frac{1}{\frac{1}{x}}$
$=x$

So $f(x)=\frac{1}{x}$ is its own inverse function.

* let $f(x) = \frac{1}{1-x}$, $g(x) = \frac{x}{x-1}$

Now $(f \circ g)(x) = f(g(x)) = f\left(\frac{x}{x-1}\right) = \frac{1}{1-\frac{x}{x-1}} = \frac{1}{\frac{x-1}{x-1} - \frac{x}{x-1}}$

$= \frac{1}{\frac{x-1-x}{x-1}} = \frac{1}{\frac{1}{x-1}} = x-1$

So $f^{-1}(x) \neq \frac{x}{x-1}$

* let $f(x) = \frac{1}{1-x}$, $g(x) = \frac{x-1}{x}$

$(f \circ g)(x) = f(g(x)) = f\left(\frac{x-1}{x}\right) = \frac{1}{1-\frac{x-1}{x}} = \frac{1}{\frac{x}{x} - \frac{x-1}{x}} = \frac{1}{\frac{x-x+1}{x}} = \frac{1}{\frac{1}{x}}$

$= x$ ✓

So $f^{-1}(x) = \frac{x-1}{x}$

* Similarly if $f(x) = \frac{x-1}{x}$ Then $f^{-1}(x) = \frac{1}{1-x}$

* let $f(x) = \frac{x}{x-1}$ $g(x) = \frac{x}{x-1}$

$(f \circ g)(x) = f\left(\frac{x}{x-1}\right) = \frac{\frac{x}{x-1}}{\frac{x}{x-1} - 1} = \frac{\frac{x}{x-1}}{\frac{x}{x-1} - \frac{x-1}{x-1}} = \frac{\frac{x}{\cancel{x-1}}}{\frac{1}{\cancel{x-1}}} = x$

So $f(x) = \dfrac{x}{x-1}$ implies $f^{-1}(x) = \dfrac{x-1}{x}$.