Connectivity in frame matroids

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Abstract

We discuss the relationship between the vertical connectivity of a biased graph Ω and the Tutte connectivity of the frame matroid of Ω (also known as the bias matroid of Ω).

1 Introduction

Throughout this paper we assume that the reader is familiar with matroid theory as in [1] and graph theory as in [6]. From the results of [5, §3] one can conclude Theorem 1.1 which characterizes the connectivity of the graphic matroid $M(G)$ in terms of the vertical connectivity and girth of G.

Theorem 1.1 (Tutte [5]). *If G is a connected graph on at least four vertices, then the connectivity of the cycle matroid $M(G)$ is exactly the minimum of the vertical connectivity of G and the girth of G.*

A *biased graph* is a pair (G, B) in which G is a graph and B is a collection of circles (i.e., simple-closed paths) in G such that no theta subgraph contains exactly two circles from B. (A *theta graph* is a subdivision of the leftmost graph in Figure 1.) The *frame matroid* of (G, B), which we denote as $M(G, B)$, is the matroid on $E(G)$ where a circuit is the edge set of either a circle from B or a subgraph that is a subdivision of one of the graphs in Figure 1 and does not contain any circles from B. Biased graphs and their matroids were first introduced by T. Zaslavsky in [9] and [11]. (In [11] the frame matroid of (G, B) is referred to as the *bias matroid* of (G, B).) A *frame matroid* M is a matroid satisfying $M = M(G, B)$ for some biased graph (G, B). Frame matroids are a generalization of Dowling geometries and their minors.

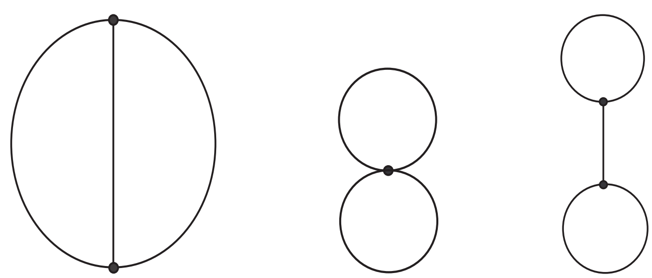

Figure 1.

Certainly there is a close relationship between the vertical connectivity of G and the connectivity of the matroid $M(G, B)$. Of course if B is the collection of all circles in G, then $M(G, B) = M(G)$ and so Theorem 1.1 gives a complete characterization of the connectivity of $M(G)$. In [2, §1.3], Pagano obtains some results relating the vertical connectivity of G with the connectivity of any general $M(G, B)$. In [7, §4], Wagner defines when a graph G is *k-biconnected* and proves that $M(G, \emptyset)$ is k-connected iff G is k-biconnected. In

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this paper we will generalize the idea of biconnectivity to all biased graphs and prove Corollary 1.2 and Theorems 1.4–1.6.

If X is a collection of edges of a graph G, then let $V(X)$ be the collection of vertices of G that are incident to edges in X and let $G{:}X$ be the subgraph of G with vertex set $V(X)$ and edge set X. For $k \geq 1$, a *k-separation* of a graph is a bipartition (X, Y) of the edges of G such that $|X| \geq k$, $|Y| \geq k$, and $|V(X) \cap V(Y)| = k$. A *vertical* k-separation of G is a k-separation (X, Y) where $V(X) \setminus V(Y) \neq \emptyset$ and $V(Y) \setminus V(X) \neq \emptyset$. A separation or vertical separation (X, Y) is said to have *connected parts* when $G{:}X$ and $G{:}Y$ are both connected.

For $k \geq 1$, a *k-biseparation* of a biased graph Ω is a bipartition (X, Y) of the edges of Ω with $|X|$ and $|Y| \geq k$ that also satisfies one of the following three properties. A subgraph of $\Omega = (G, B)$ is called *balanced* when all circles of the subgraph are contained in B, otherwise the subgraph is called *unbalanced*.

- $|V(X) \cap V(Y)| = k + 1$ and both $\Omega{:}X$ and $\Omega{:}Y$ are balanced.
- $|V(X) \cap V(Y)| = k$ and exactly one of $\Omega{:}X$ and $\Omega{:}Y$ is balanced.
- $|V(X) \cap V(Y)| = k - 1$ and both $\Omega{:}X$ and $\Omega{:}Y$ are unbalanced.

A *vertical* k-biseparation of Ω is a k-biseparation (X, Y) that has $V(X) \setminus V(Y) \neq \emptyset$ and $V(Y) \setminus V(X) \neq \emptyset$. A connected biased graph is called *k-biconnected* when it has no r-biseparation for $r < k$. A connected biased graph on at least k vertices is called *vertically k-biconnected* when it has no vertical r-biseparation for $r < k$. A biased graph is called *simple* when it contains no balanced circles of length one or two and no two unbalanced loops incident to the same vertex. Note that $M(\Omega)$ is simple iff Ω is simple. A *balancing set* of Ω is an edge set whose removal leaves a balanced subgraph.

Corollary 1.2. *If Ω is a connected and unbalanced biased graph with at least three vertices, then $M(\Omega)$ is 2-connected iff Ω is vertically 2-biconnected, has no balanced loops, and has no balancing set of rank one.*

Corollary 1.2 will be obtained from [2, Thm. 1.2.2], which is shown here in Theorem 1.3.

Theorem 1.3 (Pagano [2])**.** *If Ω is a connected biased graph but $M(\Omega)$ is not 2-connected, then there is a 1-separation of $M(\Omega)$ with connected parts.*

Theorem 1.4. *If Ω is a connected and unbalanced biased graph with at least three vertices, then $M(\Omega)$ is 3-connected iff Ω is vertically 3-biconnected, simple, and has no balancing set of rank one or two.*

Theorem 1.5. *If Ω is a connected and unbalanced biased graph with at least four vertices, then $M(\Omega)$ is vertically 4-connected iff Ω is vertically 4-biconnected and has no balancing set of rank at most three.*

A *balancing vertex* of a biased graph is a vertex contained in all unbalanced circles. Not all biased graphs have a balancing vertex and balanced biased graphs do not have balancing vertices. If Ω has a balancing vertex, then Ω does not contain two vertex-disjoint unbalanced circles. If Ω is unbalanced, does not contain two vertex-disjoint unbalanced circles, and yet does not contain a balancing vertex, then we call Ω *tangled*. The lack of vertex-disjoint unbalanced circles is a property that comes up in many areas of biased graph theory (see, for example, [2, §2.1], [3], [4], [10, Cor. 3D], and [11]).

If $M(\Omega)$ is k-connected for $k \in \{2, 3\}$, then Ω is vertically $(k-1)$-connected by Proposition 2.2. Theorem 1.6 tells us that Ω is vertically k-connected with the additional assumption that Ω is tangled.

Theorem 1.6. *If Ω is a tangled biased graph with at least four vertices and $M(\Omega)$ is k-connected for $k \in \{2, 3\}$, then Ω is vertically k-connected.*

In Section 2 we will outline some definitions and basic results that we will use in this paper, in Section 3 we will prove some lemmas, and in Section 4 we will prove our main results.

2 Definitions and basic theorems

In this section we will go over some definitions that were not presented in the introduction.

Graphs A graph G consists of a collection of vertices (i.e., topological 0-cells), denoted by $V(G)$, and a set of edges (i.e., topological 1-cells), denoted by $E(G)$, where an edge has two ends, each of which attached to a vertex. A *link* is an edge that has its ends incident to distinct vertices and a *loop* is an edge that has both of its ends incident to the same vertex.

If $X \subseteq E(G)$, then the number of vertices in $G{:}X$ is denoted by v_X and the number of components in $G{:}X$ is denoted by c_X. It is sometimes convenient to denote $E(G) \setminus X$ by $\overline{X}$.

A graph on at least $k+1$ vertices is said to be *vertically k-connected* when it is connected and there is no vertical r-separation for $r < k$. Vertical k-connectivity is usually called k-connectivity, but here we wish to distinguish between this kind of graph connectivity and the second type used in W.T. Tutte's book on graph theory ([6]).

Proposition 2.1. *If G is vertically k-connected, then every vertical k-separation has connected parts.*

Biased Graphs and Frame Matroids Given a biased graph Ω and $X \subseteq E(\Omega)$, we denote the number of balanced components of $\Omega{:}X$ by b_X. In [11, §2] it is shown that the rank function of the frame matroid $M(\Omega)$ satisfies $r(X) = v_X - b_X$ for any edge set X in Ω. We write $r(\Omega)$ as an abbreviation for $r(M(\Omega))$. If Ω has edges in at least two components, then $M(\Omega)$ is not connected.

Matroid Connectivity A *k-separation* of a matroid M is a bipartition (X, Y) of the elements of M with $|X| \geq k$ and $|Y| \geq k$ satisfying $r(A)+r(B)-r(M) \leq k-1$. M is said to be *k-connected* when there is no t-separation of M for $t < k$. M is said to be *vertically k-connected* when there is no t-separation (X, Y) of M with $t < k$, $r(X) \geq t$, and $r(Y) \geq t$. A k-separation (X, Y) is called *exact* when $r(X)+r(Y)-r(M) = k-1$. If (X, Y) is a k-separation of a frame matroid $M(\Omega)$ for which $\Omega{:}X$ and $\Omega{:}Y$ are both connected, then we say that (X, Y) has *connected parts.*

Proposition 2.2. *Let Ω be a connected and unbalanced biased graph. If (X, Y) is a vertical t-separation of Ω with connected parts, then (X, Y) is an exact k-separation of $M(\Omega)$ where $k = t+1-b_X-b_Y \in \{t-1, t, t+1\}$.*

Proof. Since (X, Y) is a t-separation of Ω,

$$r(X) + r(Y) - r(\Omega) = v_X - b_X + v_Y - b_Y - v_\Omega = t - b_X - b_Y.$$

Our conclusion now follows because $\Omega{:}X$ and $\Omega{:}Y$ are both connected and a connected biased graph on $t+1$ vertices has at least t edges when balanced and at least $t+1$ edges when unbalanced. □

Proposition 2.3. *Let Ω be a connected and unbalanced biased graph.*

(1) If (X, Y) is a k-biseparation of Ω, then (X, Y) is a k-separation of $M(\Omega)$.

(2) If (X, Y) is an exact k-separation of $M(\Omega)$ with connected parts, then (X, Y) is a k-biseparation of Ω.

3 Some Lemmas

Lemma 3.1. *Let Ω be an unbalanced biased graph. If $M(\Omega)$ is 2-connected but not 3-connected, then either*

(1) there is an exact 2-separation of $M(\Omega)$ with connected parts, or

(2) there is a minimal balancing set B of Ω such that $|B| = r(B) = 2$.

Lemma 3.2. *Let Ω be an unbalanced biased graph. If $M(\Omega)$ is 3-connected and has a vertical 3-separation, then either*

(1) there is a vertical 3-separation with connected parts or

(2) Ω has a minimal balancing set B with $r(B) = 3$.

An example of a biased graph satisfying Part (2) but not Part (1) of Lemma 3.1 is $\Omega = (K_4, B)$ in which B contains any one, two, or three quadrilaterals in K_4. Ω has no 2-biseparations, every minimal balancing set contains at least two edges, and the only minimal balancing sets of two edges are the edges outside of a balanced quadrilateral.

An example of a biased graph satisfying Part (2) but not Part (1) of Lemma 3.2 is the biased graph Ω in Figure 2 where a circle is balanced iff it contains an even number of dashed edges. There is no 3-biseparation of Ω that is a vertical 3-separation of $M(\Omega)$, every minimal balancing set contains at least three edges, and the only minimal balancing sets of three edges are the three dashed edges and the three edges parallel to the dashed edges.

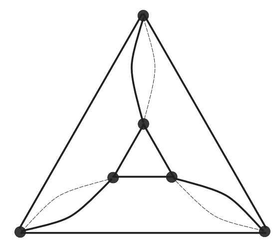

Figure 2.

Propositions 3.3–3.5 are used in the proofs of Lemmas 3.1 and 3.2.

Proposition 3.3. *Let Ω denote a biased graph and (X, Y) a bipartition with nonempty parts of $E(\Omega)$. If $X_1, \dots, X_n$ denotes the edge sets of the components of $\Omega{:}X$, then*

(1) $r(X) = \sum_{i=1}^{n} r(X_i)$ and

(2) $r(X_1) + r(\overline{X_1}) \leq r(X) + r(Y)$.

Proof. Part (1) follows directly from the form of the rank function of $M(\Omega)$. For Part (2), $r(X) + r(Y) = \sum_{i=1}^{n} r(X_i) + r(Y)$ from Part(1) and now by the submodularity property of the rank function $\sum_{i=1}^{n} r(X_i) + r(Y) \geq r(X_1) + r(Y \cup X_2 \cup \cdots \cup X_n) = r(X_1) + r(\overline{X_1})$. □

Proposition 3.4. *Let G be a connected graph and $X \subseteq E(G)$ such that $G{:}X$ is connected. If $Y \subseteq \overline{X}$ is the edge set of a connected component of $G{:}\overline{X}$, then $G{:}\overline{Y}$ is connected.*

Proposition 3.5. *If M is a matroid with an exact 3-separation (X, Y) in which $r(X)$ and $r(Y) \geq 3$, then $r(M) \geq 4$.*

Proof of Lemma 3.1. Since isolated vertices in Ω would not affect the verity of our conclusion, we may omit any isolated vertices from Ω. Thus Ω is connected because $M(\Omega)$ is connected.

Since $M(\Omega)$ is 2-connected but not 3-connected, there is an exact 2-separation (X, Y) of $M(\Omega)$. If (X, Y) has connected parts, then we are done. So say that $\Omega{:}X$ is disconnected and let $X_1, \dots, X_t$ denote the edge sets of the components of $\Omega{:}X$. Consider the bipartition $(X_i, \overline{X_i})$. Note that $|X_i| \geq 1$ and $|\overline{X_i}| \geq |Y| \geq 2$. By Proposition 3.3 we get that $r(X_i) + r(\overline{X_i}) \leq r(X) + r(Y) = r(\Omega) + 1$. When $|X_i| = 2$ this makes $(X_i, \overline{X_i})$ an exact 2-separation of $M(\Omega)$ with $\Omega{:}X_i$ connected. Either $|X_i| = 1$ for each $i \in \{1, \dots, t\}$ or there is some $|X_i| \geq 2$. Let these two possibilities be cases one and two, respectively.

<u>*Case 1:*</u> Since $|X_i| = 1$ for each $i \in \{1, \dots, t\}$ and $M(\Omega)$ is connected, each $r(X_i) = 1$. So since $\Omega{:}X_1, \dots, \Omega{:}X_t$ are pairwise vertex-disjoint, we get that X is an independent set in $M(\Omega)$. So, for any $B \subseteq X$ with $|B| = 2$ we get that $|B| = r(B) = 2$ and, using the same arguments as in the proof of Proposition 3.3,

$$r(\Omega) + 1 \leq r(B) + r(\overline{B}) \leq r(X) + r(Y) = r(\Omega) + 1$$

making $(B, \overline{B})$ an exact 2-separation of $M(\Omega)$. Furthermore, since $r(B) = 2$, we get that $r(\overline{B}) = r(\Omega) - 1$ making B a 2-element cocircuit of $M(\Omega)$. Since Ω is connected and unbalanced, the form of the rank function of $M(\Omega)$ tells us that a cocircuit is a minimal edge set whose removal leaves a biased graph with exactly one balanced component. This yields the following three possibilities: B is either a bond of Ω separating

Ω into one balanced component and one unbalanced component, one element of B is a bridge of Ω and the other a minimal balancing set on one side of the bridge, or B is a minimal balancing set of Ω.

In the first case, since the edges of B do not share any endpoints, there is a vertical 2-separation (X, Y) with connected parts, $|X|$ and $|Y| \geq 2$, one part balanced, and the other unbalanced. In either case, we have a 2-separation of $M(\Omega)$ satisfying Part (1) of our theorem. In the second case, there is a vertical 1-separation (X, Y) with connected parts, $|X|$ and $|Y| \geq 2$, and both parts unbalanced. This is a 2-separation of $M(\Omega)$ satisfying Part (1) of our theorem. In the third case, B is a minimal balancing set of Ω with $|B| = r(B) = 2$, satisfying Part (2) of our theorem.

Case 2: Without loss of generality, assume $|X_1| \geq 2$. Thus $(X_1, \overline{X_1})$ is an exact 2-separation of $M(\Omega)$ with $\Omega{:}X_1$ connected. If $\Omega{:}\overline{X_1}$ is connected, then we are done. Assuming that $\Omega{:}\overline{X_1}$ is not connected, let $Y_1, \ldots, Y_m$ denote the edge sets of the components of $\Omega{:}\overline{X_1}$. As above, either $|Y_i| = 1$ for each $i \in \{1, \ldots, m\}$ or there is some $|Y_i| \geq 2$. In the former case, as in Case 1, there is a 2-element cocircuit $B \subseteq \overline{X_1}$ which will lead to a desired 2-separation of $M(\Omega)$ or a minimal balancing set of cardinality and rank two. If there is some $|Y_i| \geq 2$, then since $\Omega{:}X_1$ is connected and Ω is connected, we get that $\Omega{:}\overline{Y}_i$ is connected by Proposition 3.4. Thus by Proposition 3.3 and the fact that $|\overline{Y_i}| \geq |X_1| \geq 2$, we get that $(Y_i, \overline{Y_i})$ is our desired 2-separation of $M(\Omega)$. □

Proof of Lemma 3.2. Since isolated vertices in Ω would not affect the verity of our conclusion, we may omit any isolated vertices from Ω. Thus Ω is connected because $M(\Omega)$ is connected.

Let (X, Y) be an exact 3-separation with $r(X)$ and $r(Y) \geq 3$. If (X, Y) has connected parts, then we are done. So assume that at least one part is not connected. Let $X_1, \ldots, X_n$ be the edge sets of the components of $\Omega{:}X$ and $Y_1, \ldots, Y_m$ be the edge sets of the components of $\Omega{:}Y$ ordered so that $r(X_1) \geq \cdots \geq r(X_n)$ and $r(Y_1) \geq \cdots \geq r(Y_m)$. Since $M(\Omega)$ is 3-connected $r(X_n), r(Y_m) \geq 1$. We now proceed in three cases: in the first case, $r(X_1)$ and $r(Y_1) \geq 3$; in the second case, either $r(X_1) \geq 3$ or $r(Y_1) \geq 3$, but not both; and in the third case, $r(X_1)$ and $r(Y_1) \leq 2$.

Case 1: Since $3 \leq r(X_1)$ and $3 \leq r(Y_1) \leq r(\overline{X_1})$ we have that $(X_1, \overline{X_1})$ is an exact 3-separation with each side of rank at least three, because Proposition 3.3 and the fact that $M(\Omega)$ is 3-connected yield

$$r(\Omega) + 2 \leq r(X_1) + r(\overline{X_1}) \leq r(X) + r(Y) = r(\Omega) + 2.$$

Now $\Omega{:}X_1$ is connected, so if $\Omega{:}\overline{X_1}$ is connected, then we are done. So suppose that $\Omega{:}\overline{X_1}$ is not connected and let $Y'_1, \ldots, Y'_t$ denote the edge sets of the components of $\Omega{:}\overline{X_1}$ numbered so that $Y_1 \subseteq Y'_1$. Thus $r(Y'_1) \geq 3$ and, since Ω and $\Omega{:}X_1$ are both connected, we get that $\Omega{:}\overline{Y'_1}$ must be connected by Proposition 3.4. So by a similar rank calculation as above we get that $(Y'_1, \overline{Y'_1})$ is an exact 3-separation of $M(\Omega)$ with connected parts and each side of rank at least three.

Case 2: Without loss of generality, $r(X_1) \geq 3$ and $r(Y_1) \leq 2$. As in Case 1, $(X_1, \overline{X_1})$ is an exact 3-separation with each side of rank at least three and $\Omega{:}X_1$ connected. If $\Omega{:}\overline{X_1}$ is connected, then we are done. If not, then let $Y'_1, \ldots, Y'_t$ denote the edge sets of the components of $\Omega{:}\overline{X_1}$ such that $r(Y'_1) \geq \cdots \geq r(Y'_t) \geq 1$. If $r(Y'_1) \geq 3$, then, as in Case 1, $(\overline{Y'_1}, Y'_1)$ is the exact 3-separation that we are looking for. So assume that $2 \geq r(Y'_1) \geq \cdots \geq r(Y'_t) \geq 1$ and let $I = V(X_1) \cap V(\overline{X_1})$. If there is some $\Omega{:}Y'_i$ that has a vertex not in I, then $r(\overline{Y'_i}) \leq r(\Omega) - 1$ which yields $r(Y'_i) + r(\overline{Y'_i}) - r(\Omega) \leq r(Y'_i) - 1$ making $(Y'_i, \overline{Y'_i})$ a 2-separation of $M(\Omega)$ when $r(Y_i) = 2$ and a 1-separation when $r(Y_i) = 1$, each a contradiction. Thus $V(\overline{X_1}) = I \subseteq V(X_1)$ and since $\Omega{:}X_1$ is connected we can conclude that $r(X_1) = r(\Omega)$ if X_1 is unbalanced and $r(X_1) = r(\Omega) - 1$ if X_1 is balanced. It must be that X_1 is balanced because if not, then $r(X_1) + r(\overline{X_1}) - r(\Omega) = 2$ yields $r(\overline{X_1}) = 2$ while we know that $r(\overline{X_1}) \geq r(Y) \geq 3$. Thus X_1 is balanced and $\overline{X_1}$ must be a balancing set for Ω. So since $r(X_1) = r(\Omega) - 1$ and $r(X_1) + r(\overline{X_1}) - r(\Omega) = 2$, it follows that $r(\overline{X_1}) = 3$. Thus there is a minimal balancing set $Y' \subseteq \overline{X_1}$ with $r(Y') \leq r(\overline{X_1}) \leq 3$. It follows that $r(Y') = 3$ because if $1 \leq r(Y') \leq 2$, then $r(\overline{Y'}) + r(Y') - r(\Omega) = r(Y') - 1$ making $(\overline{Y'}, Y')$ a 2-separation of $M(\Omega)$ when $r(Y') = 2$ and a 1-separation when $r(Y') = 1$, each a contradiction. Thus Y' is a minimal balancing set of Ω of rank three, our desired result.

Case 3: Here we have that $1 \leq r(X_i) \leq 2$ and $1 \leq r(Y_j) \leq 2$ for each i and j. So either there is some X_i or Y_j of rank one or each X_i and Y_j is of rank two. Let these be Cases 3.1 and 3.2, respectively.

Case 3.1: Without loss of generality suppose that $r(X_n) = 1$. Since Ω is connected, there is some $\Omega{:}Y_i$ that shares a vertex with $\Omega{:}X_n$. Because 3-connected matroids with at least four elements do not contain parallel elements (Ω has at least four edges by Proposition 3.5) and because we must have $|X_n \cup Y_i| \geq 2$ we get that $2 \leq r(X_n \cup Y_i) \leq r(X_n) + r(Y_i) \leq 3$. We claim that $r(X_n \cup Y_i) \neq 2$.

By way of contradiction, assume $r(X_n \cup Y_i) = 2$. Because $\Omega{:}X_1, \ldots, \Omega{:}X_n$ are pairwise vertex-disjoint and $\Omega{:}Y_1, \ldots, \Omega{:}Y_m$ are pairwise vertex-disjoint, any shared vertex of $\Omega{:}Y_i$ and $\Omega{:}X_n$ is a vertex of $\Omega{:}(X_n \cup Y_i)$ that is not contained in $\Omega{:}(\overline{X_n \cup Y_i})$. Thus $r(\overline{X_n \cup Y_i}) \leq r(\Omega) - 1$ which makes $r(X_n \cup Y_i) + r(\overline{X_n \cup Y_i}) \leq 2 + r(\Omega) - 1 = r(\Omega) + 1$ which makes $(X_n \cup Y_i, \overline{X_n \cup Y_i})$ a 2-separation of $M(\Omega)$ as long as $|\overline{X_n \cup Y_i}| \geq 2$. But since $M(\Omega)$ is 3-connected, there can be no 2-separation. Thus $|\overline{X_n \cup Y_i}| \leq 1$. However, this also makes a contradiction because now $r(\Omega) \leq r(X_n \cup Y_i) + r(\overline{X_n \cup Y_i}) \leq 3$ while Proposition 3.5 guarantees that $r(\Omega) \geq 4$. So we must have that $r(X_n \cup Y_i) = 3$.

Now, as long as $r(\overline{X_n \cup Y_i}) \geq 3$, the fact that $r(X_n \cup Y_i) = 3$ and $\Omega{:}(X_n \cup Y_i)$ is connected allows us to use the partition $(X_n \cup Y_i, \overline{X_n \cup Y_i})$ as in Case 1 or Case 2 and obtain our desired result for $M(\Omega)$. If $r(\overline{X_n \cup Y_i}) \leq 2$, then since $r(\Omega) \geq 4$ we get that $r(X_n \cup Y_i) + r(\overline{X_n \cup Y_i}) - r(\Omega) \leq 3 + r(\overline{X_n \cup Y_i}) - 4 = r(\overline{X_n \cup Y_i}) - 1$ making a 2-separation when $r(\overline{X_n \cup Y_i}) = 2$ and a 1-separation when $r(\overline{X_n \cup Y_i}) = 1$, each contradicting that $M(\Omega)$ is 3-connected.

Case 3.2: Here each X_i and Y_j is of rank two. As in Case 2 we must have that $V(X) = V(Y) = V(\Omega)$ because if there is a vertex in some $\Omega{:}X_i$ that is not in $\Omega{:}Y$, then $(X_i, \overline{X_i})$ will be a 2-separation of $M(\Omega)$, a contradiction. Similarly we reach a contradiction if there is a vertex in some $\Omega{:}Y_j$ that is not a vertex in $\Omega{:}X$. So since $r(X) + r(Y) = r(\Omega) + 2$ and $V(X) = V(Y) = V(\Omega)$ we have that

$$\begin{aligned} v_X - b_X + v_Y - b_Y &= v_\Omega + 2 \\ 2v_\Omega - (b_X + b_Y) &= v_\Omega + 2 \\ v_\Omega - 2 &= b_X + b_Y \end{aligned}$$

So either $b_X \geq \frac{v_\Omega - 2}{2}$ or $b_Y \geq \frac{v_\Omega - 2}{2}$. Without loss of generality, we may assume the former. Since each balanced $\Omega{:}X_i$ is on three vertices we also have that $3b_X \leq v_\Omega$. Combining these two inequalities we get

$$\frac{v_\Omega - 2}{2} \leq b_X \leq \frac{v_\Omega}{3}.$$

This yields that $v_\Omega \leq 6$. Given that $r(\Omega) \geq 4$ and that Ω is connected we must also have that $4 \leq v_\Omega \leq 6$. Thus we split the remainder of this case into three subcases based on the number of vertices in Ω.

Case 3.2.1: Suppose $v_\Omega = 4$. Given $\frac{v_\Omega - 2}{2} \leq b_X \leq \frac{v_\Omega}{3}$ we have $b_X = 1$. Also, since $r(X) \geq 3$ we must have that the number of components of $\Omega{:}X$ is at least two. So since one balanced component of $\Omega{:}X$ is on exactly three vertices, this leaves one vertex of Ω for the other component(s) of $\Omega{:}X$. However, a single vertex is not enough for a component of rank two, a contradiction.

Case 3.2.2: Suppose $v_\Omega = 5$. Given $\frac{v_\Omega - 2}{2} \leq b_X \leq \frac{v_\Omega}{3}$ we have $\frac{9}{6} \leq b_X \leq \frac{10}{6}$ for the integer b_X, a contradiction.

Case 3.2.3: Suppose $v_\Omega = 6$. Given $\frac{v_\Omega - 2}{2} \leq b_X \leq \frac{v_\Omega}{3}$ and $b_X + b_Y = v_\Omega - 2$ we have $b_X = b_Y = 2$. Now each balanced $\Omega{:}X_i$ and $\Omega{:}Y_j$ is on three vertices and thus can only be a balanced triangle or path of two edges without creating parallel elements or loops in $M(\Omega)$ (which 3-connected matroids with at least four elements do not have). Thus $\Omega{:}X$ and $\Omega{:}Y$ are both subgraphs of the vertex-disjoint union of two balanced triangles. Thus the underlying graph of Ω is a connected subgraph of the graph in Figure 3.

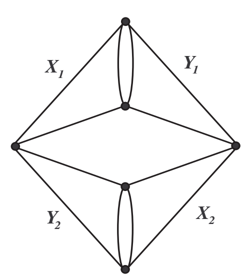

Figure 3.

Thus Ω has a vertical 2-separation $(X_1 \cup Y_1, X_2 \cup Y_2)$ with at least four elements in each part which must be a 3-separation of $M(\Omega)$ with connected parts by Proposition 2.2. This completes our proof. □

4 Proofs of our main results

Proof of Corollary 1.2. If $M(\Omega)$ is 2-connected, Proposition 2.3 implies that Ω is vertically 2-biconnected. If B is a balanced loop or balancing set of rank one, then $r(B) + r(\overline{B}) - r(\Omega) = 0$, a contradiction.

Conversely, assume that Ω has no balanced loops, no balancing set of rank one, and is vertically 2-biconnected. If $M(\Omega)$ is not 2-connected, then by Theorem 1.3, there is a 1-separation (A, B) of $M(\Omega)$ with connected parts. By Proposition 2.3, (A, B) is a 1-biseparation of Ω. However, since Ω is vertically 2-biconnected, the 1-biseparation must not be vertical. So without loss of generality, say that $V(A) = V(\Omega)$. If $\Omega{:}A$ is unbalanced, then $0 = r(A) + r(B) - r(\Omega)$ implies that $r(B) = 0$ in which case B must contain a balanced loop, a contradiction. If $\Omega{:}A$ is balanced, then B is a balancing set and $0 = r(A) + r(B) - r(\Omega)$ implies that $r(B) = 1$, a contradiction. □

Proof of Theorem 1.4. If $M(\Omega)$ is 3-connected, then Proposition 2.3 implies that Ω is vertically 3-biconnected. If Ω is not simple, then $M(\Omega)$ cannot be 3-connected because 3-connected matroids with at least four elements must be simple. (Ω has at least four edges because Ω is connected and unbalanced with at least three vertices.) If Ω has a balancing set B with $r(B) \le 2$, then $r(B) + r(\overline{B}) - r(\Omega) \le 1$, a contradiction of 3-connectedness unless $|\overline{B}| = 1$. However $|\overline{B}| = 1$ implies that $r(B) + r(\overline{B}) - r(\Omega) \le 2 + 1 - 3 = 0$, again a contradiction.

Conversely suppose that Ω is simple, has no balancing set of rank one or two, and is vertically 3-biconnected. By way of contradiction, if $M(\Omega)$ is not 3-connected, then there is a k-separation (A, B) with $k \in \{1, 2\}$. Assume that k is chosen to be a minimum. By Theorem 1.3, Lemma 3.1, and the fact that Ω has no balancing set of rank one or two, there is an exact k-separation (X, Y) of $M(\Omega)$ with connected parts which, by Proposition 2.3, is a k-biseparation of Ω. Since Ω is vertically 3-biconnected, either $V(X) = V(\Omega)$ or $V(Y) = V(\Omega)$, assume the latter. So now $r(X) + r(Y) - r(\Omega) = k - 1$ implies that $r(X) = b_Y + k - 1$ with $b_Y \in \{0, 1\}$. It cannot be that $b_Y = 1$ because then X would be a balancing set of Ω of rank one or two. So then $r(X) = k - 1 \in \{0, 1\}$ and since $|X| \ge 2$ this contradicts the simplicity of Ω. □

Proof of Theorem 1.5. Suppose that $M(\Omega)$ is vertically 4-connected. If Ω has a vertical k-biseparation (X, Y) for $k \le 3$, then by Proposition 2.3, (X, Y) is a k-separation of $M(\Omega)$. If we assume that k is a minimum, then (X, Y) also has connected parts. The definition of biseparations and the form of the rank function now imply $r(X)$ and $r(Y) \ge k$ which makes (X, Y) a vertical k-separation of $M(\Omega)$, a contradiction. Also if Ω has a balancing set B' with $r(B') \le 3$, then there is a minimal balancing set B with $r(B) = k \le 3$ and now $r(B) + r(\overline{B}) - r(\Omega) = k - 1$ which would make $(B, \overline{B})$ a vertical k-separation of $M(\Omega)$ (note that $r(\overline{B}) \ge 3$ because $v_\Omega \ge 4$ and B is a minimal balancing set which leaves $\Omega{:}\overline{B}$ connected), a contradiction.

Conversely suppose that Ω is vertically 4-biconnected and has no balancing set of rank at most three and yet $M(\Omega)$ has a vertical k-separation (X, Y) for $k \le 3$. Let Ω' be the associated simple biased graph of Ω and let $X' = X \cap \Omega'$ and $Y' = Y \cap \Omega'$. Note that (X', Y') is now a vertical t-separation of $M(\Omega')$ for $t \le 3$. Note also that Ω' is vertically 4-biconnected and does not have a balancing set of rank at most three because otherwise we then get that Ω is not vertically 4-biconnected or has a balancing set of rank at most three, a contradiction. Corollary 1.2 and Theorem 1.4 now imply that $M(\Omega')$ is 3-connected and so $t = 3$. Lemma 3.2 and the fact that there are no balancing sets of rank at most three now imply that there is a vertical 3-separation (A', B') of $M(\Omega')$ with connected parts. So if (A, B) is the associated bipartition of Ω, then (A, B) is a vertical 3-separation of $M(\Omega)$ with connected parts. Proposition 2.3 implies that (A, B) is a 3-biseparation of Ω which cannot be vertical. We will show that this leads to a contradiction and so complete our proof.

A connected biased graph of rank three has at least three vertices when unbalanced and at least four vertices when balanced. So now it cannot be that both A and B are unbalanced, because otherwise $|V(A) \cap V(B)| = 2$ implies that (A, B) is a vertical 3-biseparation of Ω, a contradiction. It cannot be that one of

A and B is balanced and the other unbalanced, because otherwise (assuming that A is balanced) $|V(A) \cap V(B)| = 3$ implies that $V(A) \setminus V(B) \neq \emptyset$ and so $V(B) \setminus V(A) = \emptyset$ making B a balancing set of rank three, a contradiction. Finally it cannot be that A and B are both balanced, because otherwise $|V(A) \cap V(B)| = 4$ and $V(A) \setminus V(B)$ or $V(B) \setminus V(A) = \emptyset$ implies that either A or B is again a balancing set of rank three, a contradiction. □

Proof of Theorem 1.6. We only do the proof for $k = 3$. The details of the proof for $k = 2$ are contained in the proof for $k = 3$.

If $M(\Omega)$ is 3-connected, then Ω is vertically 2-connected by Proposition 2.2. So by way of contradiction, assume that Ω is not vertically 3-connected. Thus there is a vertical 2-separation (A, B) of Ω where $\Omega{:}A$ and $\Omega{:}B$ are both connected and unbalanced. Let u and v be the vertices of intersection of $\Omega{:}A$ and $\Omega{:}B$. Since Ω has no balancing vertex, there is an unbalanced circle C_1 in $\Omega \backslash u$ and an unbalanced circle C_2 in $\Omega \backslash v$. Since u separates $\Omega \backslash v$ and v separates $\Omega \backslash u$, each C_i is contained entirely in one of $\Omega{:}A$ and $\Omega{:}B$. Since Ω has no two vertex-disjoint unbalanced circles, both C_1 and C_2 are contained entirely in one of $\Omega{:}A$ and $\Omega{:}B$ (say in $\Omega{:}A$). Since $\Omega{:}B$ is unbalanced, there are unbalanced circles in $\Omega{:}B$ but since there are no two vertex-disjoint unbalanced circles in Ω, every such unbalanced circle contains $\{u, v\}$. Thus u and v are both balancing vertices of $\Omega{:}B$.

In [8] it is shown that a biased graph Υ with two balancing vertices x and y has the following structure: Υ is the union of balanced subgraphs $B_1, \ldots, B_n$ (with $n \geq 2$) whose pairwise intersections are the vertices x and y and each circle intersecting two distinct B_i's is unbalanced. Thus $\Omega{:}B$ has this structure. So there is a 2-separation (B', B'') of $\Omega{:}B$ at $\{u, v\}$ where B'' is balanced and contains at least three vertices. Thus $(A \cup B', B'')$ is a vertical 2-separation of Ω with a balanced part. Thus $(A \cup B', B'')$ is a 2-separation of $M(\Omega)$ by Proposition 2.2, a contradiction of 3-connectedness. □

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