Decompositions of signed-graphic matroids

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Abstract

We give a decomposition theorem for signed graphs whose frame matroids are binary and a decomposition theorem for signed graphs whose frame matroids are quaternary.

1 Introduction

Throughout this paper we will assume that the reader is familiar with matroid theory as in [7]. The reader may or may not be familiar with signed graphs as in [17]. If not, we give an overview of all necessary information for signed graphs in Section 2.

Signed graphs and signed-graphic matroids have received and continue to receive much attention in the mathematical literature. (See, for example, [1], [3], [5], [8], [9], [11], [12], and [20].) Signed-graphic matroids have the potential to be a well-understood class of matroids much like the class of graphic matroids. It is even conjectured (in [16, §4]) that signed-graphic matroids may decompose the classes of near-regular matroids and dyadic matroids in much the same way that graphic matroids decompose the class of regular matroids in Seymour's Decomposition Theorem (see [10]). Thus more knowledge of the structure of signed-graphic matroids is desirable. One very basic matter is to understand their representability properties over various fields.

A signed graph is a pair $\Sigma = (G, \sigma)$ where G is a graph and σ is a function from the edges of G to the multiplicative group $\{+1, -1\}$. A circle (i.e., a simple closed path) in Σ is called *positive* if the product of signs on its edges is positive, otherwise the circle is called *negative*. The frame matroid of Σ (first studied by Zaslavsky in [17]) is the matroid on the edges of Σ whose circuits are edge sets of positive circles and edge sets of subgraphs that are subdivisions of the graphs shown in Figure 1 and contain no positive circles. We will call such a matroid a *signed-graphic matroid*. Signed-graphic matroids are precisely the Dowling geometries and their minors for the group of order two.

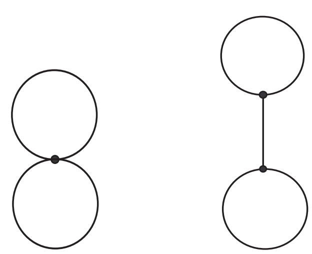

Figure 1.

Theorem 1.1 is from [17, Thm. 8B.1]. (See also Theorem 1.1 in Section 2.)

Theorem 1.1 (Zaslavsky)**.** *The matroid $M(\Sigma)$ is representable over any field of characteristic not equal to 2.*

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So it only remains to determine when $M(\Sigma)$ is representable over fields of characteristic two. It is shown in [15] that if M is representable over $GF(3)$, $\mathbb{Q}$, and a field of characteristic two, then M is representable over all fields except maybe $GF(2)$. So it only remains to find when $M(\Sigma)$ is binary (i.e., representable over $GF(2)$) and when $M(\Sigma)$ is quaternary (i.e., representable over $GF(4)$).[1]

Our main results are those shown in Sections 1.1 and 1.2. Most all of the work for proving these results is done in Gerards' monograph [5, §3.2] and Pagano's doctoral dissertation [8, Ch. 2]. In this paper, we survey and connect the pertinent information in [5] and [8] and prove some other lemmas in order to form the results of Sections 1.1 and 1.2.

The rest of this paper is organized as follows. In Section 2 we have our definitions. In Section 3 we define and discuss a notion of k-sums of signed graphs, their connection to matroid k-sums, and some applications. In Section 4 we give the proofs of our main results.

1.1 Binarity

A signed graph is called *balanced* when it has no negative circles. A *balancing vertex* in an unbalanced signed graph Σ is a vertex whose removal leaves a balanced subgraph. A signed graph is *joint unbalanced* if it is balanced after the removal of all negative loops. Negative loops are called *joints*, which is a term taken from the theory of Dowling geometries. A unbalanced signed graph is called *tangled* if it has no balancing vertex and no two vertex-disjoint negative circles.

Theorem 1.2. *If $M(\Sigma)$ and $M(\Upsilon)$ are both binary, then for each $k \in \{1,2,3\}$, $M(\Sigma \oplus_k \Upsilon) = M(\Sigma) \oplus_k M(\Upsilon)$ is binary.*

Theorem 1.3. *If Σ is connected and $M(\Sigma)$ is binary, then either*

(1) Σ is balanced,

(2) Σ is joint unbalanced,

(3) Σ has a balancing vertex,

(4) Σ is tangled, or

(5) $\Sigma = \Upsilon_1 \oplus_k \Upsilon_2$ for some $k \in \{1,2\}$ where each $M(\Upsilon_i)$ is binary.

Also, if Σ is a connected signed graph that satisfies one of (1)–(4), then $M(\Sigma)$ is binary.

Later (in Propositions 2.2 and 2.3) we show that signed graphs from Parts (1)–(3) in Theorem 1.3 have matroids that are graphic via some canonical transformations. Since the class of graphic matroids is closed under k-summing, signed graphs obtained by k-sums of the types in Parts (1)–(3) have graphic matroids. So the question of when $M(\Sigma)$ is binary and nongraphic remains. Theorem 1.4 gives a reasonable answer in the context of this paper. However the structure of a tangled signed graph is really not clear. A decomposition theorem for tangled signed graphs is given in [11]. The main result being that a tangled signed graph is obtained from a projective-planar signed graph or $-K_5$ and then a sequence of k-sums with balanced signed graphs.

Theorem 1.4. *If Σ is connected and $M(\Sigma)$ is binary, then*

(1) Σ is tangled or

(2) $M(\Sigma)$ is graphic and Σ is obtained from k-sums of signed graphs from Parts (1)–(3) in Theorem 1.3.

[1] Given Theorem 1.1 and the discussion in the introduction of [15], $M(\Sigma)$ is binary iff $M(\Sigma)$ is regular and $M(\Sigma)$ is quaternary iff $M(\Sigma)$ is near regular.

1.2 Quaternarity

The collection of joints (i.e., negative loops) of Σ is denoted by J_Σ. A jointless signed graph is called *cylindrical* if it may be imbedded in the plane with at most two negative faces. The signed graph T_6 is shown in Figure 2 where a solid edge is positive and a dashed edge is negative.

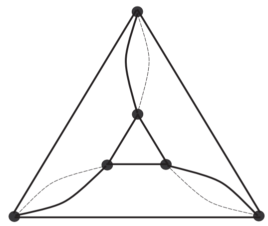

Figure 2.

Theorem 1.5. *If $M(\Sigma)$ and $M(\Upsilon)$ are both quaternary, then for each $k \in \{1,2,3\}$, $M(\Sigma \oplus_k \Upsilon) = M(\Sigma) \oplus_k M(\Upsilon)$ is quaternary.*

Theorem 1.6. *If Σ is connected and $M(\Sigma)$ is quaternary, then either*

(1) $M(\Sigma)$ is binary,

(2) $\Sigma \setminus J_\Sigma$ has a balancing vertex,

(3) $\Sigma \setminus J_\Sigma$ is cylindrical,

(4) $\Sigma \setminus J_\Sigma \cong T_6$, or

(5) $\Sigma \setminus J_\Sigma = \Upsilon_1 \oplus_i \Upsilon_2$ for $i \in \{1,2,3\}$ where each $M(\Upsilon_i)$ is quaternary.

Also, if Σ is a connected signed graph that satisfies one of (1)–(4), then $M(\Sigma)$ is quaternary.

Since joints in a signed graph have some effect on quaternarity, we give Theorem 1.8 that tells us when we can add and remove joints without affecting quaternarity. A special fact about tangled signed graphs that is used in the conclusion of Theorem 1.8 is Proposition 1.7. The proof of Proposition 1.7 is straightforward and is left to the reader.

Proposition 1.7. *If Σ is tangled, then Σ has has exactly one unbalanced block (in particular, Σ will be jointless).*

Theorem 1.8. *If Σ is connected and $M(\Sigma)$ is quaternary, then either*

(1) Σ is tangled and has no joints or

(2) Σ is not tangled, joints may be added and removed from Σ without effecting quaternarity, and $\Sigma \setminus J_\Sigma$ is obtained by k-sums of signed graphs that: are balanced after removing any joints, have balancing vertices after removing any joints, are isomorphic to T_6 after removing any joints, and are cylindrical after removing any joints.

2 Preliminaries

Graphs A graph G consists of a collection of vertices (i.e., topological 0-cells), denoted by $V(G)$, and a set of edges (i.e., topological 1-cells), denoted by $E(G)$, where an edge has two ends each of which attached to a vertex. A *link* is an edge that has its ends incident to distinct vertices and a *loop* is an edge that has both of its ends incident to the same vertex.

A *circle* is a connected, 2-regular graph (i.e., a simple closed path). In graph theory a circle is often called a cycle, circuit, polygon, etc. We denote the cycle matroid of the graph G by $M(G)$. If $X \subseteq E(G)$, then we denote the subgraph of G consisting of the edges in X and all vertices incident to an edge in X by $G{:}X$. The collection of vertices in $G{:}X$ is denoted by $V(X)$, the number of vertices in $G{:}X$ is denoted by v_X, and the number of connected components in $G{:}X$ is denoted by c_X.

For $k \geq 1$, a *k-separation* of a graph is a bipartition (A, B) of the edges of G such that $|A| \geq k$, $|B| \geq k$, and $|V(A) \cap V(B)| = k$. A *vertical* k-separation (A, B) of G is a k-separation where $V(A) \setminus V(B) \neq \emptyset$ and $V(B) \setminus V(A) \neq \emptyset$. A separation or vertical separation (A, B) is said to have *connected parts* when $G{:}A$ and $G{:}B$ are both connected. A connected graph on at least $k + 1$ vertices is said to be *vertically k-connected* when there is no vertical r-separation for $r < k$. Vertical k-connectivity is usually called k-connectivity, but here we wish to distinguish between this kind of graph connectivity and the second type used in Tutte's book on graph theory ([14]).

Signed graphs A *signed graph* is a pair (G, σ) in which $\sigma : E(G) \to \{+1, -1\}$. A circle or path in a signed graph Σ is called *positive* if the product of signs on its edges is positive, otherwise the circle or path is called *negative*. A negative loop is often called a *joint* which is a widely used term in the study of Dowling geometries. If H is a subgraph of Σ, then H is called *balanced* when all circles in H are positive. A *balancing vertex* of an unbalanced signed graph is a vertex whose removal leaves a balanced subgraph. Not all unbalanced signed graphs have balancing vertices and balanced signed graphs do not have balancing vertices. When drawing signed graphs, positive edges are represented by solid curves and negative edges by dashed curves. We consider a graph G to be a signed graph with all edges signed positively. In this sense, the class of signed graphs contains the class of graphs.

A *switching function* on a signed graph $\Sigma = (G, \sigma)$ is a function $\eta : V(\Sigma) \to \{+1, -1\}$. The signed graph $\Sigma^\eta = (G, \sigma^\eta)$ has sign function σ^η defined on all edges of G by $\sigma^\eta(e) = \eta(v)\sigma(e)\eta(w)$ where v and w are the endpoint vertices (or endpoint vertex) of edge e. The signed graphs Σ and Σ^η have the same list of positive circles. When two signed graphs Σ_1 and Σ_2 satisfy $\Sigma_1^\eta = \Sigma_2$ for some switching function η, the two signed graphs are said to be *switching equivalent*. An important notion in the study of signed graphs is that two signed graphs with the same underlying graph are switching equivalent iff they have the same list of positive circles (see [17, Prop. 3.2]). Switching equivalent signed graphs are considered to be isomorphic.

In a signed graph $\Sigma = (G, \sigma)$, the deletion of e from Σ is defined as $\Sigma \backslash e = (G \backslash e, \sigma)$ where σ is restricted to the domain $E(G \backslash e)$. The contraction of an edge e is defined for three distinct cases. If e is a link, then $\Sigma / e = (G/e, \sigma^\eta)$ where η is a switching function satisfying $\sigma^\eta(e) = +$, which always exists. Note that the contraction Σ/e is only well defined up to switching. If e is a positive loop, then $\Sigma/e = \Sigma \backslash e$. If e is a joint, then Σ/e is the signed graph obtained from Σ as follows: links incident to v become joints incident to their other endpoint, loops incident to v become positive loops incident to v, and edges not incident to v remain unchanged. The reason for this definition of contraction in signed graphs is so that contractions in signed graphs will correspond to contractions in their signed-graphic matroids.

A *minor* of Σ is a signed graph obtained from Σ by a sequence of contractions and deletions of edges, deletions of isolated vertices, and switchings. A *link minor* of Σ is a minor obtained without contracting any joints.

A signed graph is called *tangled* if it is unbalanced, has no balancing vertex, and no two vertex-disjoint negative circles.

Signed-graphic matroids The central topic in this work is called the *frame matroid* of a signed graph (also called the *bias matroid* of a signed graph in [19]). Since this is the main topic, we will call the frame matroid of a signed graph a *signed-graphic matroid*. We denote the signed-graphic matroid of Σ by $M(\Sigma)$. The element set of $M(\Sigma)$ is $E(\Sigma)$ and circuit of $M(\Sigma)$ is either the edge set of a positive circle or the edge set of a subdivision of a subgraph in Figure 1 with no positive circles.

With the definition of deletions and contractions of signed graphs above, for any $e \in E(\Sigma)$, we have that $M(\Sigma \backslash e) = M(\Sigma) \backslash e$ and $M(\Sigma/e) = M(\Sigma)/e$ (see [17, Thm. 5.2]). It is important to note that if $\Sigma = (G, \sigma)$ is balanced, then $M(\Sigma) = M(G)$. In this sense, the class of signed-graphic matroids properly contains the class of graphic matroids. Given two signed graphs Σ_1 and Σ_2 with the same underlying graph, $M(\Sigma_1) \cong M(\Sigma_2)$ iff Σ_1 and Σ_2 have the same lists of positive circles iff Σ_1 is switching equivalent to Σ_2.

Given $X \subseteq E(\Sigma)$ we denote the number of balanced connected components of $\Sigma{:}X$ by b_X. If $X \subseteq E(\Sigma)$, then $r(X) = v_X - b_X$ (see [17, Thm. 5.1(j)]). For brevity we write $r(\Sigma)$ to mean $r(M(\Sigma))$. If a signed graph Σ is not connected and has no isolated vertices, then $M(\Sigma)$ is not connected.

Given a signed graph $\Sigma = (G, \sigma)$ we construct the *incidence matrix* $I(\Sigma)$ as follows. Let the columns of $I(\Sigma)$ be indexed by $E(\Sigma)$ and the rows by $V(\Sigma)$. The column in $I(\Sigma)$ corresponding to $e \in E(\Sigma)$ has the following form: if e is a positive loop then the column is zero; if e is a joint with endpoint v then the column has a nonzero entry in the row corresponding to v and zero in all other rows; if e is a link with endpoints u and v then the column has a 1 in the row corresponding to u, $-\sigma(e)$ in the row corresponding v, and zero in all other rows. Theorem 1.1 is from [17, Thm. 8B.1].

Theorem 2.1 (Zaslavsky)**.** *The matroid of $I(\Sigma)$ over any field of characteristic other than two is $M(\Sigma)$.*

Joint-unbalanced signed graphs A signed graph Σ is called *joint unbalanced* if its only negative circles are loops. Let G be the ordinary graph obtained from Σ by adding a new vertex v and replacing all joints of Σ with links from the joint endpoint to v.

Proposition 2.2. *If Σ is joint-unbalanced, then the graph G obtained as above satisfies $M(G) = M(\Sigma)$.*

Proof. Let Υ be the signed graph with underlying graph G and all edges signed positive. If e is a new joint added to Υ with endpoint v, then $(\Upsilon \cup e)/e = \Sigma$ up to switching. Also, e is a coloop in $M(\Upsilon \cup e)$ and so $M(G) = M(\Upsilon) = M(\Upsilon \cup e)\backslash e = M(\Upsilon \cup e)/e = M((\Upsilon \cup e)/e) = M(\Sigma)$. □

Balancing vertices in signed graphs Let Σ have a balancing vertex v. By sign switching we may assume that all negative links of Σ are incident to v. Let G be the ordinary graph obtained by splitting v into two vertices v_+ and v_- where positive links incident to v are incident to v_+, negative links incident to v are incident to v_-, and joints incident to v become links between v_+ and v_-.

Proposition 2.3. *If Σ has a balancing vertex, then the graph G obtained as above satisfies $M(G) = M(\Sigma)$.*

Proof. Let Υ be the signed graph with underlying graph G and all edges signed positive. If e is a negative link added to Υ with endpoints v_+ and v_-, then $(\Upsilon \cup e)/e = \Sigma$ up to switching. Also, e is a coloop in $M(\Upsilon \cup e)$ and so $M(G) = M(\Upsilon) = M(\Upsilon \cup e)\backslash e = M(\Upsilon \cup e)/e = M((\Upsilon \cup e)/e) = M(\Sigma)$. □

Proposition 2.4 is a special case of [18, Cor. 2]. It is a fact we will need in several proofs.

Proposition 2.4. *If Σ is connected and has two balancing vertices x and y, then there is a bipartition (A, B) of $E(\Sigma)$ with $V(A) \cap V(B) = \{x, y\}$ such that $\Sigma{:}A$ and $\Sigma{:}B$ are both connected and balanced.*

Previous results on binarity The 4-point line is the matroid of rank two on four elements whose circuits are all subsets of order three. In [13], Tutte shows that a matroid is binary iff it does not contain the 4-point line as a minor. The only signed graph whose matroid is the 4-point line is the signed graph shown in Figure 3. This signed graph is called the *4-edge line*.

Figure 3.


In [8, Thm 2.0.6], the following characterization of signed graphs whose matroids are binary is given. Its proof is based on Tutte's excluded-minor characterization of binary matroids.

Theorem 2.5 (Pagano [8])**.** *Given a connected signed graph Σ, let $\overline{\Sigma}$ be the signed graph obtained from Σ by contracting all balanced blocks. Then $M(\Sigma)$ is binary iff $\overline{\Sigma}$ has no two vertex-disjoint negative circles.*

Theorem 2.6 is a simple corollary of Theorem 2.5 and it motivates the definition of a tangled signed graph.

Theorem 2.6. *If Σ is vertically 2-connected, is unbalanced, has no balancing vertex, and is not joint unbalanced, then $M(\Sigma)$ is binary iff Σ is a tangled signed graph.*

3 k-sums of signed graphs.

1-sums Let Σ and Υ be signed graphs with nonempty edge sets such that Υ is balanced. The *1-sum* of Σ and Υ is the identification of Σ and Υ along some vertex. Proposition 3.1 is immediate from our definition of a signed-graphic 1-sum and the definition of a matroid 1-sum.

Proposition 3.1. *If Σ and Υ are signed graphs, then $M(\Sigma \oplus_1 \Upsilon) = M(\Sigma) \oplus_1 M(\Upsilon)$.*

2-sums Given two signed graphs Σ and Υ we will define two methods of taking their 2-sum. By $\Sigma \oplus_2 \Upsilon$ we mean a 2-sum that is one of these two types. If both of Σ and Υ are unbalanced, then the *1-vertex 2-sum* is obtained by identifying the signed graphs along a joint and then deleting the joint. If exactly one of Σ and Υ is unbalanced, then the *2-vertex 2-sum* of the signed graphs is obtained by choosing a link in each signed graph, switching so that the links have the same sign in each, identifying the two signed graphs along the links, and then deleting that link. In both cases it is required that the edge along which the 2-sum is taken is not a coloop in the signed-graphic matroid. The verification of Proposition 3.2 is routine and we leave it to the reader.

Proposition 3.2. *If Σ and Υ are signed graphs, then $M(\Sigma \oplus_2 \Upsilon) = M(\Sigma) \oplus_2 M(\Upsilon)$.*

3-sums Given two signed graphs Σ and Υ we will define two methods of taking their 3-sum. By $\Sigma \oplus_3 \Upsilon$ we mean a 3-sum that is one of these two types. In all cases we require that each term of a 3-sum has matroid rank at least three. If Σ and Υ are both unbalanced, then their *2-vertex 3-sum* is obtained by identifying the signed graphs along a 4-edge line in each (see Figure 3) and then deleting the edges of the line. If exactly one of Σ and Υ is unbalanced, then their *3-vertex 3-sum* is obtained by selecting a positive triangle in each, switching so that the edges have the same sign pattern in each triangle, identifying the signed graphs along the triangle, and then deleting the edges of the triangle.

In order to relate this signed-graphic 3-sum to matroid 3-sums we will use the notion of modular sums from [2]. Given two matroids M and N both containing a line L that is modular in at least one of M and N, we define the 3-sum $M \oplus_3 N$ as the modular sum of M and N along L. Now let A and B be matrices, over some field $\mathbb{F}$ with a common submatrix M_L of rank two as shown.

$$A = \left(\begin{array}{c|c} A_1 & \mathbf{0} \\ \hline A_2 & M_L \end{array}\right) \qquad B = \left(\begin{array}{c|c} M_L & B_1 \\ \hline \mathbf{0} & B_2 \end{array}\right)$$

Use $A \oplus_3 B$ to denote the matrix obtained by identifying A and B along M_L and then deleting the columns of M_L. From [2, Theorem 6.12] we have Proposition 3.3.

Proposition 3.3. *If $A \oplus_3 B$ is a matrix 3-sum defined over a submatrix L, then as long as L is a line in $M(A)$ and a modular line in $M(B)$, then $M(A \oplus_3 B) = M(A) \oplus_3 M(B)$.*

Note that a positive triangle in Σ and a 4-edge line in Σ are both modular lines in $M(\Sigma)$. So now using Proposition 3.3, the definitions of 3-sums and incidence matrices, and Theorem 2.1 we get Proposition 3.4.

Proposition 3.4. *If Σ and Υ are signed graphs, then*

(1) after scaling columns (if necessary) $I(\Sigma) \oplus_3 I(\Upsilon) = I(\Sigma \oplus_3 \Upsilon)$ and

(2) $M(\Sigma \oplus_3 \Upsilon) = M(\Sigma) \oplus_3 M(\Upsilon)$.

3.1 Some results of Gerards and Pagano that use k-sums

Now that we have defined k-sums of signed graphs we will discuss the results of Gerards [5] and Pagano [8] that we will use in proving Theorem 1.6. Theorem 3.5 (from [8, Ch. 2]) is a partial result towards Theorem 1.6.

Theorem 3.5 (Pagano)**.** *Let Σ be a vertically 3-connected signed graph without loops such that $M(\Sigma)$ is not binary. If $M(\Sigma)$ is quaternary, then Σ*

(1) is cylindrical,

(2) is isomorphic to T_6, or

(3) is a 3-vertex 3-sum of Υ_1 and Υ_2 where $M(\Upsilon_1)$ is quaternary and not binary.

The proof of Theorem 3.5 in [8] is long and it uses Theorem 3.6, which is a forbidden-minor characterization of the signed graphs whose matroids are quaternary. Theorem 3.6 is proven in [8, §2.2] simply by considering the complete list of seven forbidden minors for quaternary matroids found by Geelen, Gerards, and Kapoor in [4] and checking which signed graphs have frame matroids equal to one of these forbidden minors. Denote the three signed graphs in Figure 4 by $\pm C_3^{(1)}$, $\pm C_4 \backslash e$, and $-K_4^{(1)}$, respectively. The matroid $M(\pm C_3^{(1)})$ is the nonfano plane and $M(\pm C_4 \backslash e)$ and $M(-K_4^{(1)})$ are both the dual of the nonfano plane.

Theorem 3.6 (Pagano)**.** *A signed-graphic matroid $M(\Sigma)$ is quaternary iff Σ does not contain any of the signed graphs in Figure 4 as a minor.*

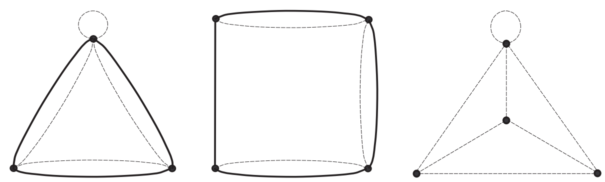

Figure 4.

The signed graphs $-K_4$ and $\pm C_3$ are shown in Figure 5, respectively from left to right. In [5, §3.2], Gerards studies the class of signed graphs containing no $-K_4$ nor $\pm C_3$ link minor.

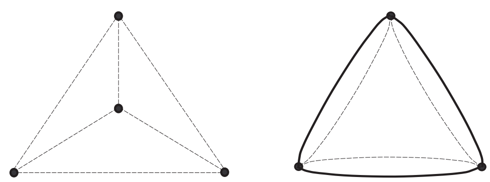

Figure 5.

Clearly there is a close relationship between the class of signed graphs with a minor from Figure 4 and the class of signed graphs with a link minor from Figure 5. Theorem 3.7 reveals that these classes are exactly the same for connected signed graphs with at least one joint.

Theorem 3.7. *If Σ is a connected and contains at least one joint, then the following are equivalent.*

(1) Σ contains one of $\pm C_3^{(1)}$, $\pm C_4 \backslash e$, and $-K_4^{(1)}$ as a minor.

(2) Σ contains $-K_4$ or $\pm C_3$ as a minor.

(3) $\Sigma \setminus J_\Sigma$ contains $-K_4$ or $\pm C_3$ as a link minor.

Lemma 3.8. *Let Υ be a signed graph without positive loops. If Σ contains Υ as a minor, then $\Sigma \setminus J_\Sigma$ contains $\Upsilon \setminus J_\Upsilon$ as a link minor.*

Proof. Let $\widetilde{\Sigma}$ denote a minimal subgraph of Σ that contracts to Υ and let C be a collection of edges in $\widetilde{\Sigma}$ such that $\widetilde{\Sigma}/C = \Upsilon$. Let $T \subseteq C$ be the edges of a maximal forest of $\widetilde{\Sigma}{:}C$. Contracting the edges of a tree all at once is accomplished by switching the edges of the tree to be all positive, removing the edges of the tree, and then coalescing the vertices of the tree to a single vertex. So now the edges of $C \setminus T$ in $\widetilde{\Sigma}/T$ are all loops and, since $\widetilde{\Sigma}$ is minimal, these loops are all negative.

Write $C \setminus T = \{e_1, \ldots, e_n\}$ and let v_i be the endpoint vertex of e_i in $\widetilde{\Sigma}/T$. Since Υ contains no positive loops, the edges in $\widetilde{\Sigma}/T$ incident to v_i besides the ones in $C \setminus T$ are all links. So now, contracting $e_1, \ldots, e_n$ all at once is accomplished by removing $e_1, \ldots, e_n$ and making all of the remaining edges incident to some v_i joints incident to their other endpoints. Thus $\Upsilon \setminus J_\Upsilon$ may be obtained as a deletion of $\widetilde{\Sigma}/T$ rather than as a contraction. Thus $\Upsilon \setminus J_\Upsilon$ may be obtained as a link minor of $\Sigma \setminus J_\Sigma$. □

Proof of Theorem 3.7. (1)→(2) Let Υ be a such a minor of Σ. By Lemma 3.8, $\Sigma \setminus J_\Sigma$ contains $\Upsilon \setminus J_\Upsilon$ as a minor. Our desired conclusion now follows.

(2)→(3) Immediate from Lemma 3.8 because $\pm C_3$ and $-K_4$ are jointless.

(3)→(1) Let $\widetilde{\Sigma}$ be a minimal subgraph of $\Sigma \setminus J_\Sigma$ that contracts by links to $\pm C_3$ or $-K_4$ and let $e \in J_\Sigma$. Since Σ is connected, there is a path γ connecting e to $\widetilde{\Sigma}$. Since $\pm C_3$ and $-K_4$ are both connected and $\widetilde{\Sigma}$ minimal, $\widetilde{\Sigma}$ is connected. So now $\widetilde{\Sigma} \cup \gamma \cup e$ contracts to $\pm C_3^{(1)}$ or $-K_4^{(1)}$. □

Theorems 3.9 and 3.10 are from [5, §3.2]. They are reworded here to use our notion of k-sums rather than the slightly different notion of k-splits presented therein. These results give a complete decomposition and construction method for connected signed graphs containing neither $\pm C_3$ nor $-K_4$ as a link minor. So by Theorem 3.7 this is a complete structure theorem for connected signed graphs with joints whose matroids are quaternary. So our task in this paper is really to find the jointless signed graphs that have quaternary matroids and have $\pm C_3$ or $-K_4$ as a minor. Say that a k-sum $\Sigma \oplus_k \Upsilon$ is *minimal* if $k = 1$ or $k \in \{2, 3\}$ and $\Sigma \oplus_k \Upsilon$ cannot be expressed as a t-sum of two other signed graphs for some $t < k$.

Theorem 3.9 (Gerards)**.** *If $\Sigma \oplus_k \Upsilon$ is jointless and a minimal k-sum, then $\Sigma \oplus_k \Upsilon$ contains either $-K_4$ or $\pm C_3$ as a link minor iff one of Σ and Υ contains either $-K_4$ or $\pm C_3$ as a link minor.*

Theorem 3.10 (Gerards)**.** *If Σ is a connected signed graph containing neither $-K_4$ nor $\pm C_3$ as a link minor, then $\Sigma \setminus J_\Sigma$ either*

(1) is balanced,

(2) has a balancing vertex,

(3) is cylindrical,

(4) is isomorphic to T_6,

(5) is $\Upsilon \oplus_k \Omega$ for some $k \in \{1, 2, 3\}$.

An important observation in conjunction with Theorem 3.10 is Proposition 3.11.

Proposition 3.11. *If Σ is a connected signed graph such that $\Sigma \setminus J_\Sigma$ is balanced, has a balancing vertex, is cylindrical, or is isomorphic to T_6, then Σ has neither $-K_4$ nor $\pm C_3$ as a link minor.*

Proof. By Theorem 3.7 we need only show that $\Sigma \setminus J_\Sigma$ has neither $-K_4$ nor $\pm C_3$ as a link minor. Certainly signed graphs that are balanced or have balancing vertices do not contain these link minors because both of these minors are unbalanced and do not have balancing vertices. Also T_6 does not contain one of these link minors by inspection. The class of cylindrical signed graphs is closed under taking link minors and one can check that neither $-K_4$ nor $\pm C_3$ is cylindrical. Thus a cylindrical signed graph contains neither of these link minors. □

3.2 k-sums and quaternarity

Proposition 3.12 is a technical result that we will need for Theorem 1.5 whose proof is immediately below Proposition 3.12. Write $GF(4) = \mathbb{Z}_2[x]/(x^2 + x + 1) = \{0, 1, \omega, 1 + \omega\}$ in which ω is a root of $x^2 + x + 1$. The proof of Proposition 3.12 is left to the reader. It uses Proposition 3.3 and the fact that there is an automorphism of $GF(4)$ that takes ω to $\omega + 1$.

Proposition 3.12. *If M and N are quaternary matroids intersecting at a common k-point line, then $M \oplus_3 N$ is quaternary.*

Proof of Theorem 1.5. Propositions 3.1, 3.2, and 3.4 imply that $M(\Sigma \oplus_k \Upsilon) = M(\Sigma) \oplus_k M(\Upsilon)$. So Proposition 3.12 and the fact that 1-sums and 2-sums preserve representability over a field imply that $M(\Sigma) \oplus_k M(\Upsilon)$ is quaternary. □

Conversely to Theorem 1.5, assume that $M(\Sigma \oplus_k \Upsilon) = M(\Sigma) \oplus_k M(\Upsilon)$ is $\mathbb{F}$-representable. If $k \in \{1, 2\}$, then it follows that both $M(\Sigma)$ and $M(\Upsilon)$ are $\mathbb{F}$-representable because each term is a minor of the sum. However, if $k = 3$ then we cannot guarantee that either of $M(\Sigma)$ and $M(\Upsilon)$ is a minor of $M(\Sigma \oplus_3 \Upsilon)$, even if $M(\Sigma \oplus_3 \Upsilon)$ is 3-connected and the 3-sum is minimal.

Theorem 3.13. *Say that $\Sigma \oplus_3 \Upsilon$ is jointless and a minimal 3-sum. If $M(\Sigma \oplus_3 \Upsilon)$ is quaternary, then $M(\Sigma)$ and $M(\Upsilon)$ are both quaternary.*

Lemma 3.14. *If $\Sigma \oplus_3 \Upsilon$ is a 3-vertex 3-sum that is minimal and has balanced term Υ, then either Σ is a minor of $\Sigma \oplus_3 \Upsilon$ or $\Upsilon \cong K_4$.*

Lemma 3.14 can be proven as a consequence of [10, 4.2], but here we present a direct graph-theoretical proof in order to make the presentation more self contained.

Proof of Lemma 3.14. Let T be the triangle along which the 3-sum is taken. By minimality $M(\Sigma \oplus_3 \Upsilon)$ has no loops or parallel elements. So any positive circle in $\Sigma \oplus_3 \Upsilon$ must have length at least three. We now proceed in two cases. In the first case, there is a circle in $\Upsilon \setminus E(T)$ and, in the second case, there is no circle in $\Upsilon \setminus E(T)$.

Case 1: Let C be a circle in $\Upsilon \setminus E(T)$. Thus C is balanced and so has length at least three. So either there are three disjoint paths in $\Upsilon \setminus E(T)$ connecting the circle C to $V(T)$ or, by Menger's Theorem there is a vertex set S in Υ with $1 \leq |S| \leq 2$ that separates C from Σ. The latter case, however, is not possible because then there would be a vertical 1 or 2-separation of $\Sigma \oplus_3 \Upsilon$ with one part balanced. In either case we have a contradiction to the assumption that the 3-sum is minimal. Thus there are three disjoint paths, call them γ_1, γ_2 and γ_3, connecting C to $V(T)$. So now $(\Sigma \setminus E(T)) \cup \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup C$ is contained in $\Sigma \oplus_3 \Upsilon$ and contracts to Σ.

Case 2: Since $r(\Upsilon) \geq 3$ with 3-sums, there is a vertex v in $\Upsilon \setminus V(T)$. Now either there are three internally-disjoint paths, call them γ_1, γ_2, and γ_3, connecting v to $V(T)$ or, by Menger's Theorem, there is a vertex set S in Υ with $1 \leq |S| \leq 2$ that separates v from Σ. The latter case, again, is not possible as in *Case 1.* Now either $\gamma_1 \cup \gamma_2 \cup \gamma_3$ contains all of the edges of $\Upsilon \setminus E(T)$ or there is some other edge f outside of $\gamma_1 \cup \gamma_2 \cup \gamma_3$.

In the former case we must have that each γ_i is a path of length one, or else there is a vertex of degree two in $\Sigma \oplus_3 \Upsilon$. But a vertex of degree 2 would create a vertical 2-separation of $\Sigma \oplus_3 \Upsilon$ with a balanced part, a contradiction of the minimality of our 3-sum. Thus $\Upsilon = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup T \cong K_4$, as required.

In the latter case, since Υ is balanced and $M(\Sigma \oplus_3 \Upsilon)$ is simple, the edge f must be a link. Since $\Sigma \oplus_3 \Upsilon$ must be vertically 2-connected when the 3-sum is minimal Menger's theorem implies that there are disjoint paths α_1 and α_2 in $\Upsilon \setminus E(T)$ connecting the endpoints of f to $\gamma_1 \cup \gamma_2 \cup \gamma_3$. However now we have that $\gamma_1 \cup \gamma_2 \cup \gamma_3 \cup f \cup \alpha_1 \cup \alpha_2$ contains a circle and is contained in $\Upsilon \setminus E(T)$, a contradiction of our assumption of no circles in $\Upsilon \setminus E(T)$. □

Lemma 3.15. *$M(\Sigma)$ is quaternary iff $M(\Sigma \oplus_3 K_4)$ is quaternary.*

Proof. If $M(\Sigma)$ is quaternary, then $M(\Sigma \oplus_3 K_4)$ is quaternary from Theorem 1.5 or from the more general result [6, Lemma 3.2]. Now if $M(\Sigma \oplus_3 K_4)$ is quaternary it follows from [6, Lemma 2.11], the fact that $\mathbb{F}$-representability is closed under duality, and again by [6, Lemma 3.2] that $M(\Sigma)$ is quaternary. □

Proof of Theorem 3.13. The proof for 3-vertex 3-sums follows from Lemmas 3.14 and 3.15 and the fact that the class of $\mathbb{F}$-representable matroids is minor closed.

Now assume that $\Sigma \oplus_3 \Upsilon$ is a 2-vertex 3-sum over the 4-edge line L. Without loss of generality we need only conclude that $M(\Sigma)$ is quaternary. We will proceed by contradiction and assume that $M(\Sigma)$ is not quaternary while $M(\Sigma \oplus_3 \Upsilon)$ is quaternary. The contradiction we will arrive at is that $\Sigma \oplus_3 \Upsilon$ contains one of $\pm C_3^{(1)}$, $\pm C_4 \backslash e$, and $-K_4^{(1)}$ as a minor.

First, we claim that there is a negative circle N in $\Upsilon \setminus E(L)$ that intersects at most one vertex of $V(L)$. By minimality of the k-sum, $\Sigma \oplus_3 \Upsilon$ is vertically 2-connected and every vertical 2-separation has both parts unbalanced. Thus $\Upsilon \setminus E(L)$ is unbalanced and there is a negative circle N in $\Upsilon \setminus E(L)$. Since there are no joints, N must have length at least two. Furthermore, we may assume that N contains at most one vertex from $V(L)$ unless both vertices of $V(L)$ are balancing vertices of $\Upsilon \setminus E(L)$. However, if both vertices are balancing vertices of $\Upsilon \setminus E(L)$, then by Proposition 2.4 there is a partition (A, B) of the edges of $\Upsilon \setminus E(L)$ with both parts balanced and $V(A) \cap V(B) = V(L)$. Furthermore, since $r(\Upsilon) \geq 3$ is required for 3-sums, either $v_A \geq 3$ or $v_B \geq 3$ (assume that $v_A \geq 3$). Thus $(E(\Sigma \setminus E(L)) \cup B, A)$ is a vertical 2-separation of $\Sigma \oplus_3 \Upsilon$ with one part balanced, a contradiction of minimality. Thus N exists.

Now because Σ contains at least two joints and $M(\Sigma)$ is not quaternary, Theorem 3.7 implies that $\Sigma \setminus J_\Sigma$ contains a $\pm C_3$ or $-K_4$ link minor. Let $\widetilde{\Sigma}$ denote a minimal subgraph of $\Sigma \setminus J_\Sigma$ that contracts by links to one of $\pm C_3$ and $-K_4$. Since both $\pm C_3$ and $-K_4$ are connected and $\widetilde{\Sigma}$ is minimal, $\widetilde{\Sigma}$ is connected. Let C denote an edge set in $\widetilde{\Sigma}$ such that $\widetilde{\Sigma}/C$ is either $\pm C_3$ or $-K_4$. Since we are only contracting links, each component of $\Sigma{:}C$ is balanced. We finish the proof in three cases. In the first case $\widetilde{\Sigma}$ may be chosen so that it does not intersect $E(L)$, in the second case $\widetilde{\Sigma}$ must intersect $E(L)$ but one of the edges of $E(L)$ is contracted to obtain the desired minor, and in the third case $\widetilde{\Sigma}$ must intersect $E(L)$ and no edge of $E(L)$ is contracted to obtain the desired minor.

Case 1: Since $\widetilde{\Sigma} \cap E(L) = \emptyset$, $\widetilde{\Sigma}$ is a subgraph of $\Sigma \setminus E(L)$ which is a subgraph of $\Sigma \oplus_3 \Upsilon$. So since $\Sigma \oplus_3 \Upsilon$ is vertically 2-connected, there is a path γ in $\Sigma \oplus_3 \Upsilon$ connecting the negative circle N in $\Upsilon \setminus E(L)$ to $\widetilde{\Sigma}$. So now $\widetilde{\Sigma} \cup \gamma \cup N$ contracts to one of $\pm C_3^{(1)}$ and $-K_4^{(1)}$, a contradiction.

Case 2: Since $\widetilde{\Sigma}$ is contained in $\Sigma \setminus J_\Sigma$, the only common edges of $\widetilde{\Sigma}$ and L are links. Let e be one of these common links and say e is contracted to obtain the desired minor. Thus $\widetilde{\Sigma}/e$ contracts to either $\pm C_3$ or $-K_4$ and $\widetilde{\Sigma}/e$ is a minor of $\Sigma \setminus J_\Sigma$. Also, if f is the other link of L, then f is a joint in $(\widetilde{\Sigma} \cup f)/e$. So since $\widetilde{\Sigma}$ is connected, $(\widetilde{\Sigma} \cup f)/e$ is connected and so $(\widetilde{\Sigma} \cup f)/e$ contracts to one of $\pm C_3^{(1)}$ or $-K_4^{(1)}$. But $(\widetilde{\Sigma} \cup f)/e$ is a minor of $\Sigma \setminus J_\Sigma$ and, as we will show in the next paragraph, $\Sigma \setminus J_\Sigma$ is a minor of $\Sigma \oplus_3 \Upsilon$. Thus one of $\pm C_3^{(1)}$ and $-K_4^{(1)}$ is a minor of $\Sigma \oplus_3 \Upsilon$, a contradiction.

Since $\Sigma \oplus_3 \Upsilon$ is vertically 2-connected there are two disjoint paths γ_1 and γ_2 in $\Upsilon \setminus E(L)$ that connect N to the vertices of L. So now $(\Sigma \setminus E(L)) \cup \gamma_1 \cup \gamma_2 \cup N$ is contained in $\Sigma \oplus_3 \Upsilon$ and contracts to $\Sigma \setminus J_\Sigma$.

Case 3: Let e and f denote the links of L. Without loss of generality we can say that $e \in \widetilde{\Sigma}$. By the hypothesis of this case e and $f \notin C$. So now C is contained in $\widetilde{\Sigma} \setminus \{e, f\}$ which is contained in $\Sigma \setminus E(L)$ which is contained in $\Sigma \oplus_3 \Upsilon$ and $\widetilde{\Sigma} \setminus \{e, f\}$ contains both vertices of L. Now since $\Sigma \oplus_3 \Upsilon$ is vertically 2-connected, there are disjoint paths γ_1 and γ_2 in $\Upsilon \setminus E(L)$ connecting the negative circle N to $V(L)$. Recall that N contains at most one vertex of L. Now by inspection $(\widetilde{\Sigma}/C) \setminus \{e, f\} \cup \gamma_1 \cup \gamma_2 \cup N$ contracts to $\pm C_4 \setminus e$ or $-K_4^{(1)}$, a contradiction. □

3.3 k-sums and binarity

Proof of Theorem 1.2. Propositions 3.1, 3.2, and 3.4 imply that $M(\Sigma \oplus_k \Upsilon) = M(\Sigma) \oplus_k M(\Upsilon)$. So the fact that the class of binary matroids is closed under modular summing implies that $M(\Sigma) \oplus_k M(\Upsilon)$ is binary. □

As in the previous section, if $M(\Sigma \oplus_k \Upsilon)$ is $\mathbb{F}$-representable and $k \in \{1, 2\}$, then $M(\Sigma)$ and $M(\Upsilon)$ are both $\mathbb{F}$-representable. Since our main result on binary signed-graphic matroids (Theorem 1.3) only uses 1-sums and 2-sums, this will be enough for our purposes.

3.4 k-sums and tangledness

In this section we will prove Theorem 3.16 which is necessary for Theorem 1.8.

Theorem 3.16. *If Σ is tangled, then Σ contains $-K_4$ or $\pm C_3$ as a link minor.*

Lemma 3.17. *Let $\Sigma \oplus_k \Upsilon$ be a k-vertex k-sum that is a minimal k-sum. If Σ is unbalanced and Υ is balanced, then Σ is tangled iff $\Sigma \oplus_k \Upsilon$ is tangled.*

Proof. The result is evident for $k = 1$. The proof for $k = 2$ is an easier version of the proof for $k = 3$ so we will only do the proof for $k = 3$. Let T be the triangle along which the sum is taken. We first claim that Υ is vertically 3-connected. Suppose by way of contradiction that there is a vertical 2-separation of Υ. Since T is a triangle, $V(T)$ lies completely in one part of this separation. Thus we can create a vertical 2-separation of $\Sigma \oplus_3 \Upsilon$ with one part balanced, a contradiction of the minimality of the k-sum.

We will show that Σ has two vertex-disjoint negative circles iff $\Sigma \oplus_3 \Upsilon$ has two vertex-disjoint negative circles and that Σ has a balancing vertex iff $\Sigma \oplus_3 \Upsilon$ has a balancing vertex. This will prove our result.

Let C_1 and C_2 be vertex-disjoint negative circles in Σ. Since T has three vertices, at most one of C_1 and C_2 may intersect $E(T)$. If neither C_1 nor C_2 intersects $E(T)$, then C_1 and C_2 are vertex-disjoint negative circles in $\Sigma \oplus_3 \Upsilon$, as required. If, say, C_1 intersects $E(T)$, then because Υ is vertically 3-connected, there is a path γ in $\Upsilon \setminus E(T)$ that connects the endpoints of $C_1 \setminus E(T)$ and avoids the third vertex of T. Thus $(C_1 \setminus E(T)) \cup \gamma$ and C_2 are vertex-disjoint negative circles in $\Sigma \oplus_3 \Upsilon$, as required.

Conversely, say C_1 and C_2 are vertex-disjoint negative circles in $\Sigma \oplus_3 \Upsilon$. Since Υ is balanced, at most one of C_1 and C_2 intersects $E(\Upsilon) \setminus E(T)$. If neither intersects Υ, then C_1 and C_2 are vertex-disjoint negative circles in Σ. If, say, C_1 intersects $E(\Upsilon) \setminus E(T)$, then this intersection is a path with endpoints in T. So if e is the link in T with these two endpoints, then $(C_1 \setminus E(\Upsilon)) \cup e$ and C_2 are vertex-disjoint negative circles in Σ.

Now let v be a balancing vertex of Σ, thus all negative circles of Σ intersect v. Since Υ is balanced, any negative circle in $\Sigma \oplus_3 \Upsilon$ which has a non-empty intersection with $E(\Upsilon) \setminus E(T)$ is obtained as a symmetric difference of a negative circle in Σ and a circle in Υ. Thus every negative circle of $\Sigma \oplus_3 \Upsilon$ intersects v, so v is a balancing vertex of $\Sigma \oplus_3 \Upsilon$.

Conversely let v be a balancing vertex of $\Sigma \oplus_3 \Upsilon$. If v is not a balancing vertex of Σ, then there is a negative circle N in $\Sigma \backslash v$. As before we can use N and the vertical 3-connectivity of Υ to construct a negative circle in $\Sigma \oplus_3 \Upsilon$ that avoids v, a contradiction. Thus v is a balancing vertex of Σ. □

Lemma 3.18. *If Σ is a signed graph and (A, B) is a vertical t-separation with $t \in \{1, 2, 3\}$ such that both parts are balanced, then Σ is balanced or has a balancing vertex.*

Proof. The conclusion for $t \in \{1, 2\}$ is evident, so say $t = 3$. Let η be a switching function on $\Sigma{:}A$ that makes all edges positive and let ξ be a switching function on $\Sigma{:}B$ that makes all edges positive. By replacing ξ with $-\xi$ if necessary, we may assume that η and ξ disagree on at most one of the three vertices of $(\Sigma{:}A) \cap (\Sigma{:}B)$. If they agree on all vertices, then Σ is balanced. If they disagree on one vertex, then that vertex is a balancing vertex of Σ. □

Lemma 3.19. *If Σ is tangled and vertically 2-connected and (X, Y) is a vertical 2-separation with both parts unbalanced, then either there is a bipartition (Y_1, Y_2) of Y such that Y_2 is balanced and $(X \cup Y_1, Y_2)$ is a vertical 2-separation of Σ or there is a bipartition (X_1, X_2) of X satisfying the corresponding condition.*

Proof. Let u and v be the vertices of $V(X) \cap V(Y)$. Since Σ has no balancing vertex, $\Sigma \backslash u$ contains a negative circle C_1 and $\Sigma \backslash v$ contains a negative circle C_2. Each C_i must then be contained entirely in $\Sigma{:}X$ or entirely in $\Sigma{:}Y$. Furthermore, it cannot be that one of C_1 and C_2 is contained in $\Sigma{:}X$ and the other is in $\Sigma{:}Y$ because then C_1 and C_2 would be vertex disjoint, a contradiction. So without loss of generality C_1 and C_2 are both contained in $\Sigma{:}X$. Since Y is unbalanced, it contains negative circles and since Σ is tangled, each negative circle must contain both u and v. Thus u and v are both balancing vertices of $\Sigma{:}Y$.

Proposition 2.4 now implies that there is a bipartition (Y_1, Y_2) of Y such that $V(Y_1) \cap V(Y_2) = \{u, v\}$ and each Y_i is balanced. Since $v_Y \geq 3$ we have that $v_{Y_1} \geq 3$ or $v_{Y_2} \geq 3$ (assume the latter). So now $(X \cup Y_1, Y_2)$ is our desired vertical 2-separation. □

Lemma 3.20. *A vertically 3-connected cylindrical signed graph without a balancing vertex is not tangled.*

Proof. Let Σ be a vertically 3-connected signed graph imbedded in the plane with two negative faces. Since Σ is vertically 3-connected, each facial boundary is a circle and the intersection of two facial boundaries is either empty, a single vertex, or an edge. Thus Σ has a balancing vertex iff the two negative facial boundaries intersect in a path or vertex. So if Σ has no balancing vertex, then it has two vertex-disjoint negative circles and so is not tangled. $\square$

Proof of Theorem 3.16. Note that a tangled signed graph must have at least three vertices. If Σ is a tangled signed graph on three or four vertices, then it is easy to check that Σ has $\pm C_3$ as a minor or $-K_4$ as a subgraph. Now consider a tangled signed graph on $n \geq 5$ vertices. By Proposition 1.7 we may also assume that Σ is loopless and Σ has no two parallel links with the same sign. Assume by way of contradiction that Σ has neither $-K_4$ nor $\pm C_3$ as a link minor. Because Σ is tangled, it is unbalanced and has no balancing vertex. Furthermore, it cannot be that $\Sigma \cong T_6$ because T_6 is not tangled and Σ is not cylindrical and vertically 3-connected by Lemma 3.20. Thus by Theorems 3.9 and 3.10 we have that $\Sigma = \Upsilon_1 \oplus_k \Upsilon_2$ for some $k \in \{1, 2, 3\}$ where each Υ_i has neither $-K_4$ nor $\pm C_3$ as a link minor and $|V(\Upsilon_i)| < |V(\Sigma)|$. Furthermore, we may assume that this k-sum is minimal.

Now let U_i be the edges of Υ_i in Σ. It must be that at least one of U_1 and U_2 is unbalanced by Lemma 3.18. If both U_1 and U_2 are unbalanced, then the k-sum is either a 1-vertex 2-sum or a 2-vertex 3-sum. The former case is not possible because Σ only has one unbalanced block by Proposition 1.7. The latter case is not possible because if it were, then by Lemma 3.19, we would express the 2-vertex 3-sum as a 2-vertex 2-sum which contradicts the minimality of k. Thus exactly one of U_1 and U_2 is balanced and so the k-sum is a k-vertex k-sum. Say that Υ_1 is unbalanced and Υ_2 is balanced. Now Υ_1 is tangled by Lemma 3.17. By induction on the number of vertices, we get that Υ_1 contains a $\pm C_3$ or $-K_4$ link minor. But then Σ contains a $\pm C_3$ or $-K_4$ link minor by Theorem 3.9, a contradiction. $\square$

4 Remaining proofs

Proof of Theorem 1.3. If Σ is balanced, then Σ satisfies (1). So assume that Σ is unbalanced. If $|V(\Sigma)| \leq 2$, then since $M(\Sigma)$ is binary, Σ does not contain a 4-edge line and so Σ must either have a balancing vertex (satisfying (3)) or be joint unbalanced (satisfying (2)). So now assume that $|V(\Sigma)| \geq 3$. If Σ is vertically 2-connected, then by Theorem 2.6, Σ is either tangled (satisfying (4)), has a balancing vertex (satisfying (3)), or is joint unbalanced (satisfying (2)). If Σ is connected but not vertically 2-connected, then since $|V(\Sigma)| \geq 3$ there is a vertical 1-separation of Σ with connected parts. So we may write Σ as a 1-sum or 1-vertex 2-sum $\Upsilon_1 \oplus_i \Upsilon_2$. Since the class of $\mathbb{F}$-representable matroids is minor closed, each $M(\Upsilon_i)$ is binary. Thus Σ satisfies (5).

The concluding part of the theorem follows from the fact that graphic matroids are binary, Propositions 2.2 and 2.3, and Theorem 2.5. $\square$

Proof of Theorem 1.4. Decompose Σ into terms given in Theorem 1.3. If none of the terms are tangled, then all terms in the sum have graphic matroids (from Propositions 2.2 and 2.3). Since the class of graphic matroids is closed under k-sums, $M(\Sigma)$ is graphic and satisfies (2). If there is a tangled term in the sum, then since tangled signed graphs are jointless (by Proposition 1.7) each sum in the construction starting from the tangled term is a 1-sum and so we get that Σ is tangled. $\square$

Proof of Theorem 1.6. If Σ is balanced, then Σ satisfies (1). So assume that Σ is unbalanced. If Σ has a joint, then by Theorems 3.7, 3.9, and 3.10, Σ satisfies our conclusion. So now assume that Σ is jointless.

If $|V(\Sigma)| \leq 3$, then Σ must have a balancing vertex (and so Σ satisfies (2)) unless Σ contains $\pm C_3$ as a subgraph. In the latter case, however, $M(\Sigma)$ is binary because Σ is tangled and so Σ satisfies (1). So now assume that $|V(\Sigma)| \geq 4$.

If Σ is vertically 3-connected, then Σ satisfies our conclusion by Theorem 3.5. If Σ is vertically 2-connected, then Σ has a vertical 2-separation with a balanced part or every vertical 2-separation has both parts unbalanced. In the former case, we can write Σ as a 2-vertex 2-sum and in the latter case we can write Σ as a 2-vertex 3-sum that is a minimal 3-sum. By Theorem 3.13 and the fact that $\mathbb{F}$-representability is closed under taking minors, Σ satisfies (5). If Σ is not vertically 2-connected, then we can write Σ as a 1-vertex 2-sum or 1-sum and again Σ satisfies (5).

The final statement of the theorem follows from Proposition 3.11, Theorems 3.7 and 3.6, and the fact that binary matroids are also quaternary. □

Proof of Theorem 1.8. If $\Sigma \setminus J_\Sigma$ is tangled, then Σ has no joints and is tangled by Theorems 3.16, 3.7, and 3.6. Thus Σ satisfies (1). So suppose that $\Sigma \setminus J_\Sigma$ is not tangled. We will show that Σ satisfies (2). Decompose $\Sigma \setminus J_\Sigma$ into terms according to Theorem 1.6 and then further decompose the resulting binary terms according to Theorem 1.3. Assume that each sum in the decomposition is minimal. If there is a term in the decomposition that is tangled, then since tangled signed graphs are jointless, each sum in the construction of $\Sigma \setminus J_\Sigma$ starting from the tangled term is a 1-sum or k-vertex k-sum. Thus $\Sigma \setminus J_\Sigma$ is tangled by Lemma 3.17, a contradiction. Thus each term in the decomposition after removing any joints either: is balanced, has a balancing vertex, is isomorphic to T_6, or is cylindrical. By Proposition 3.11 and Theorem 3.7, no term has a $\pm C_3$ or $-K_4$ link minor. Thus $\Sigma \setminus J_\Sigma$ does not have a $\pm C_3$ or $-K_4$ link minor by Theorems 3.9 and 3.7. Thus joints may be added or removed from $\Sigma \setminus J_\Sigma$ and Σ without affecting quaternerity by Theorem 3.7. Thus Σ satisfies (2). □

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